Three-particle scattering amplitudes from lattice QCD

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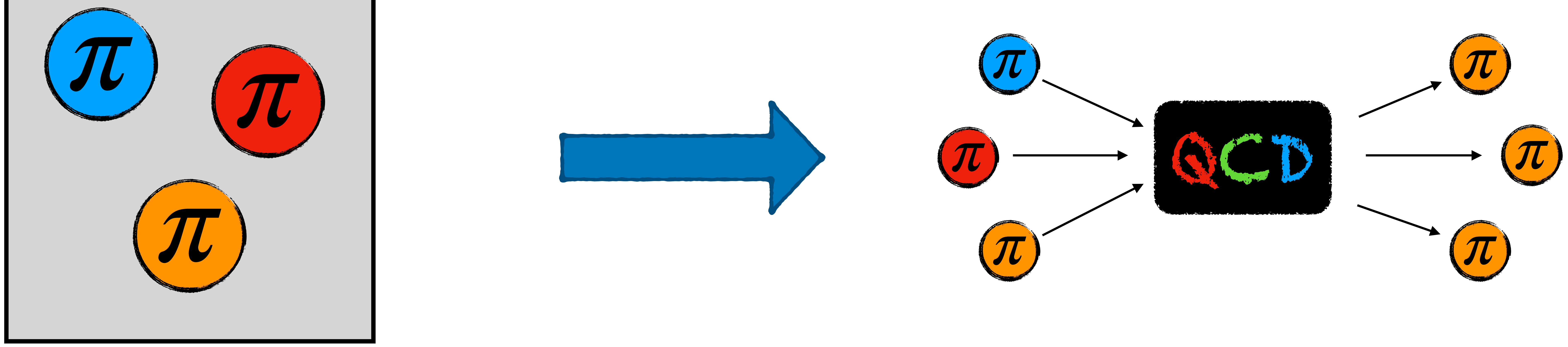
Valencia, 15th Dec



Acknowledgements

IFIC people

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Three-particle people

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Carsten Urbach

Introduction

Quantum Chromodynamics

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  - CP violating decays: $K \to \pi\pi,\ K \to \pi\pi\pi, D \to \pi\pi, K\bar{K}, (\pi\pi\pi\pi), \ldots$

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- First-principles nuclear interactions.

Towards the QCD S-Matrix

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

Lattice QCD $\longrightarrow$ QCD S-matrix

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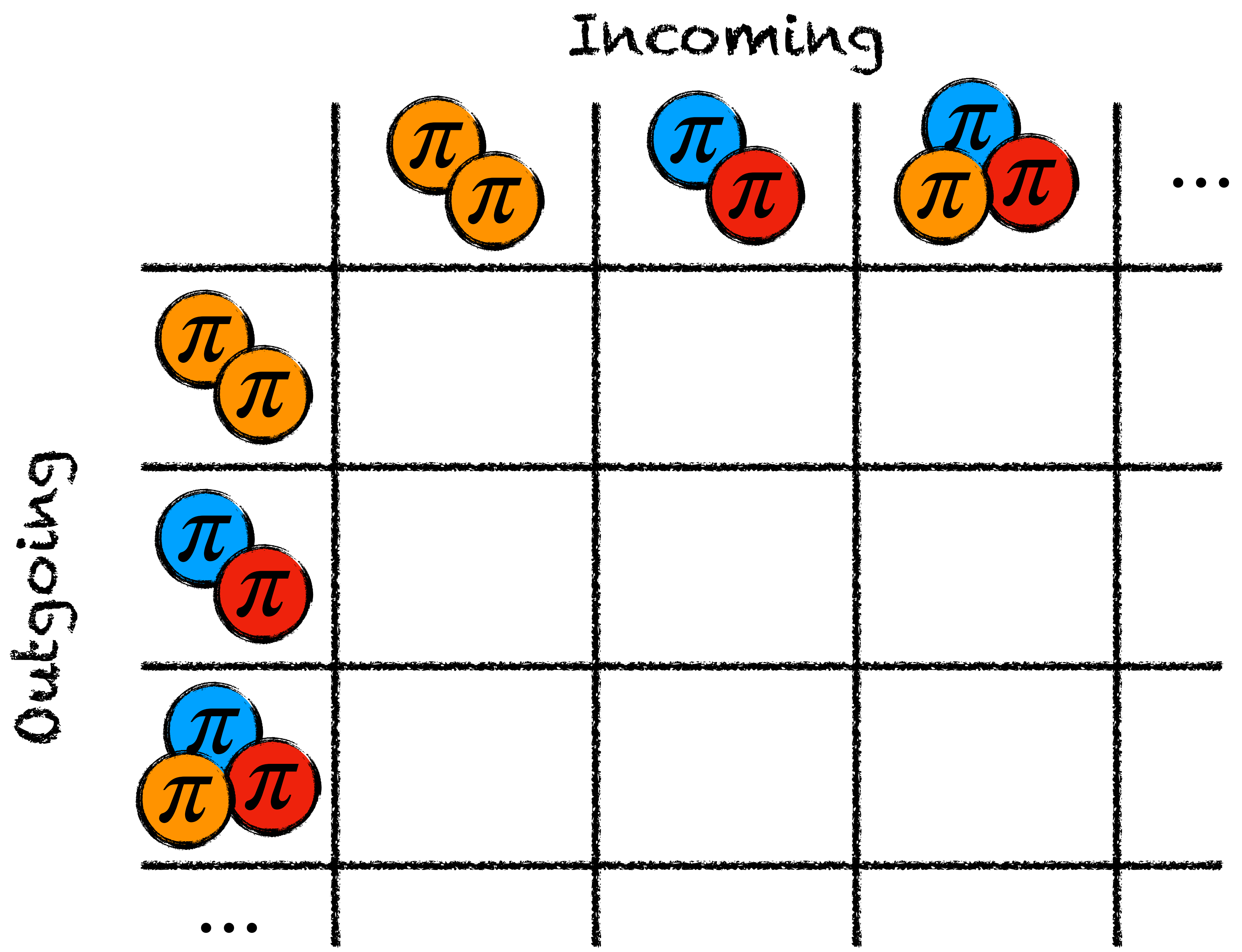

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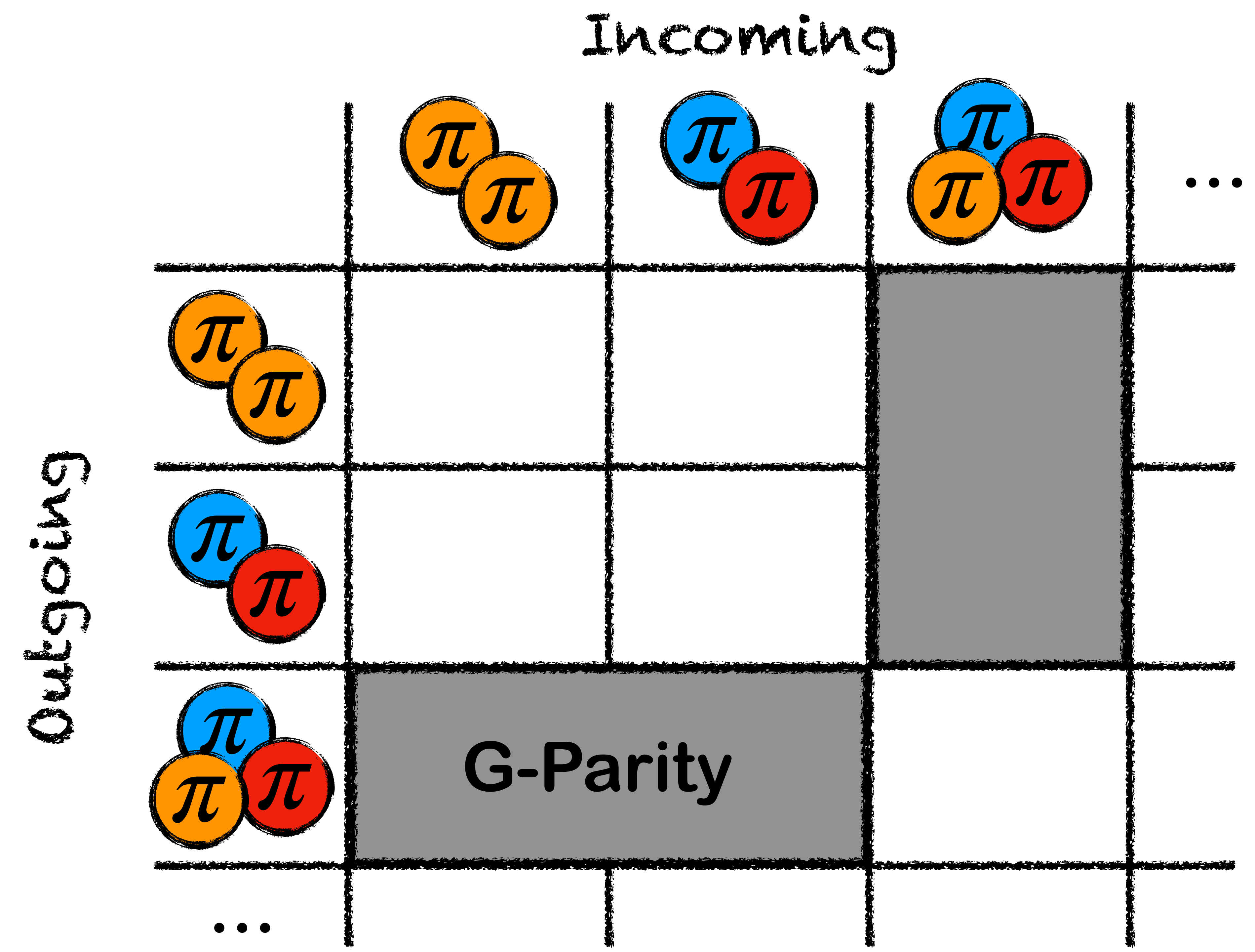

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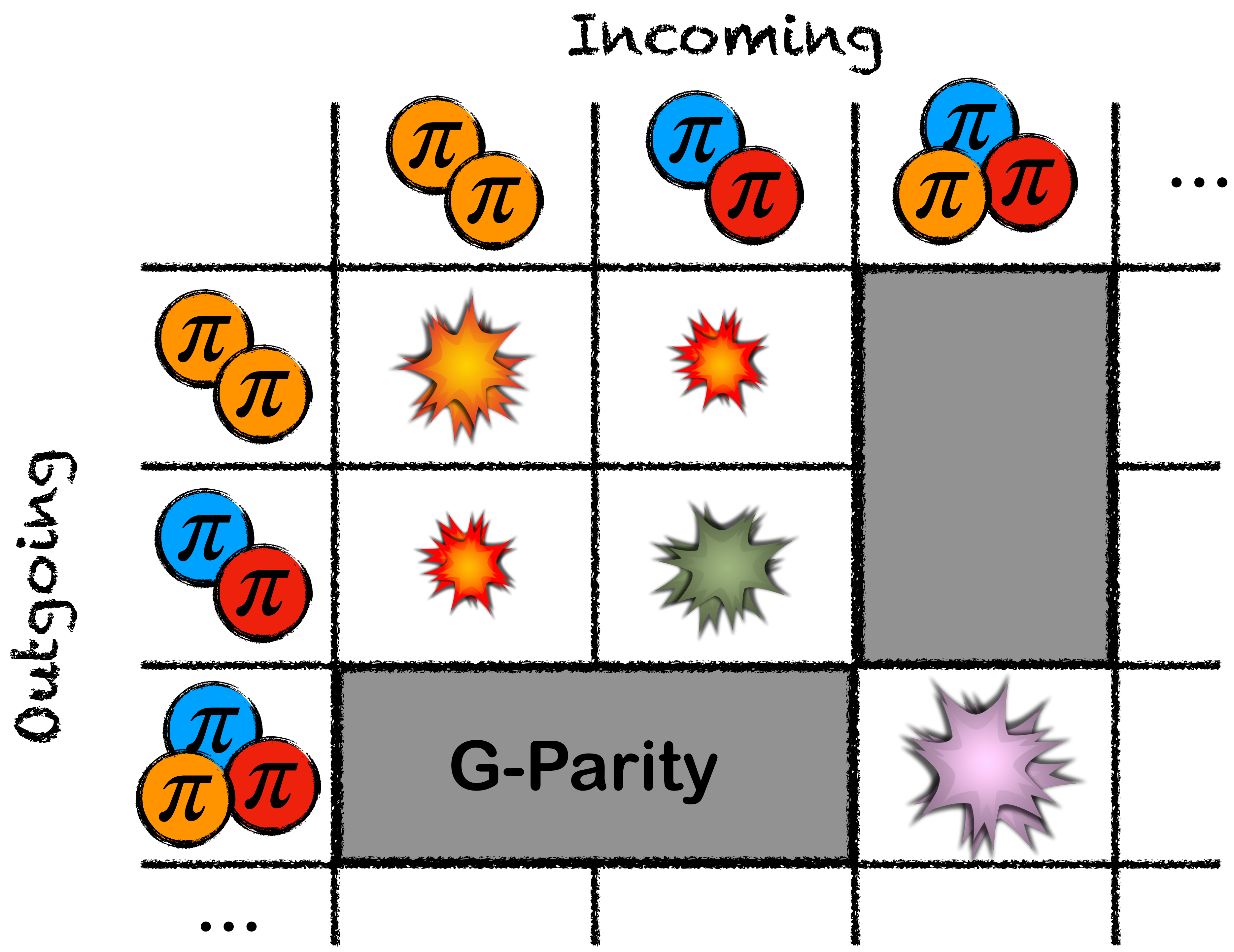

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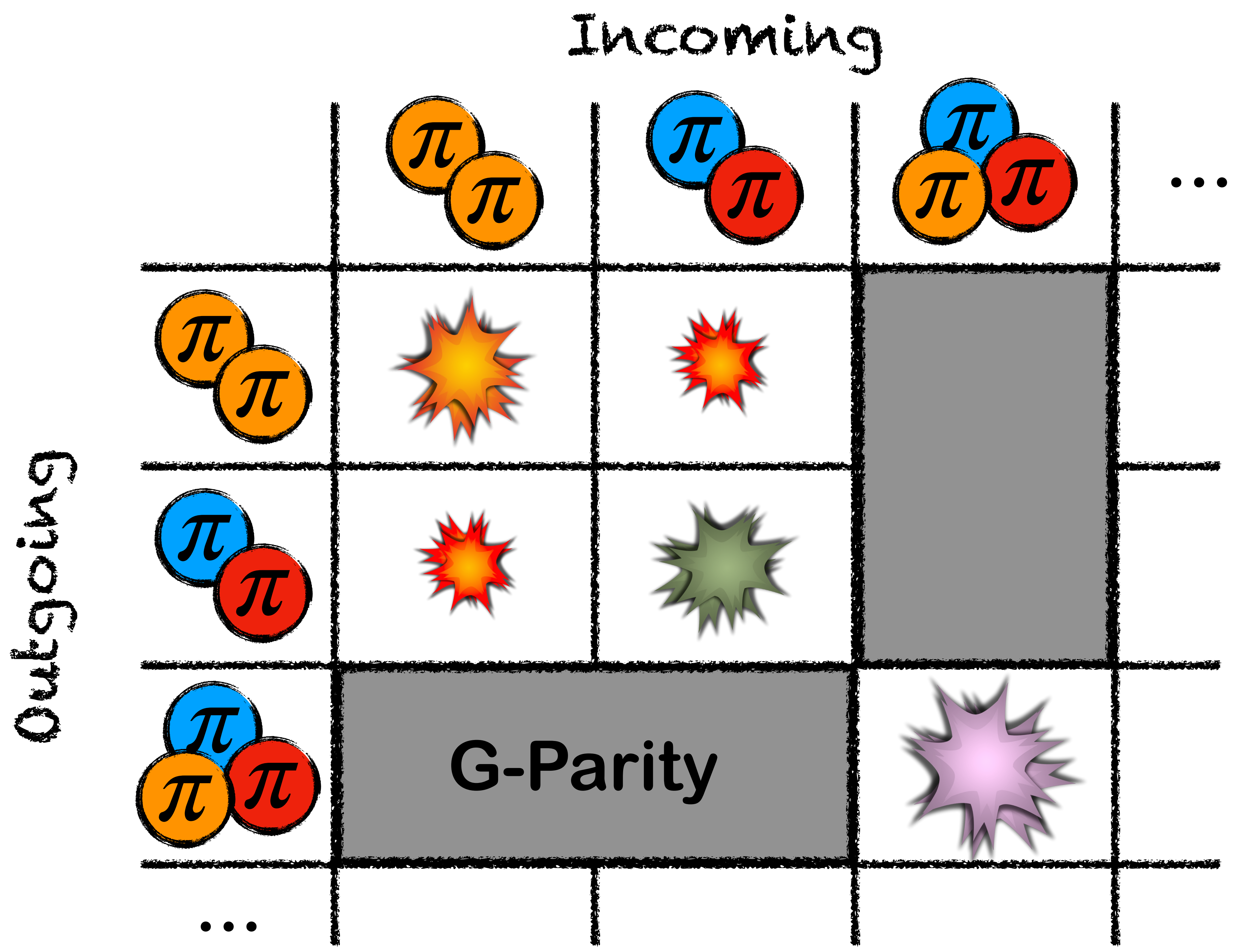
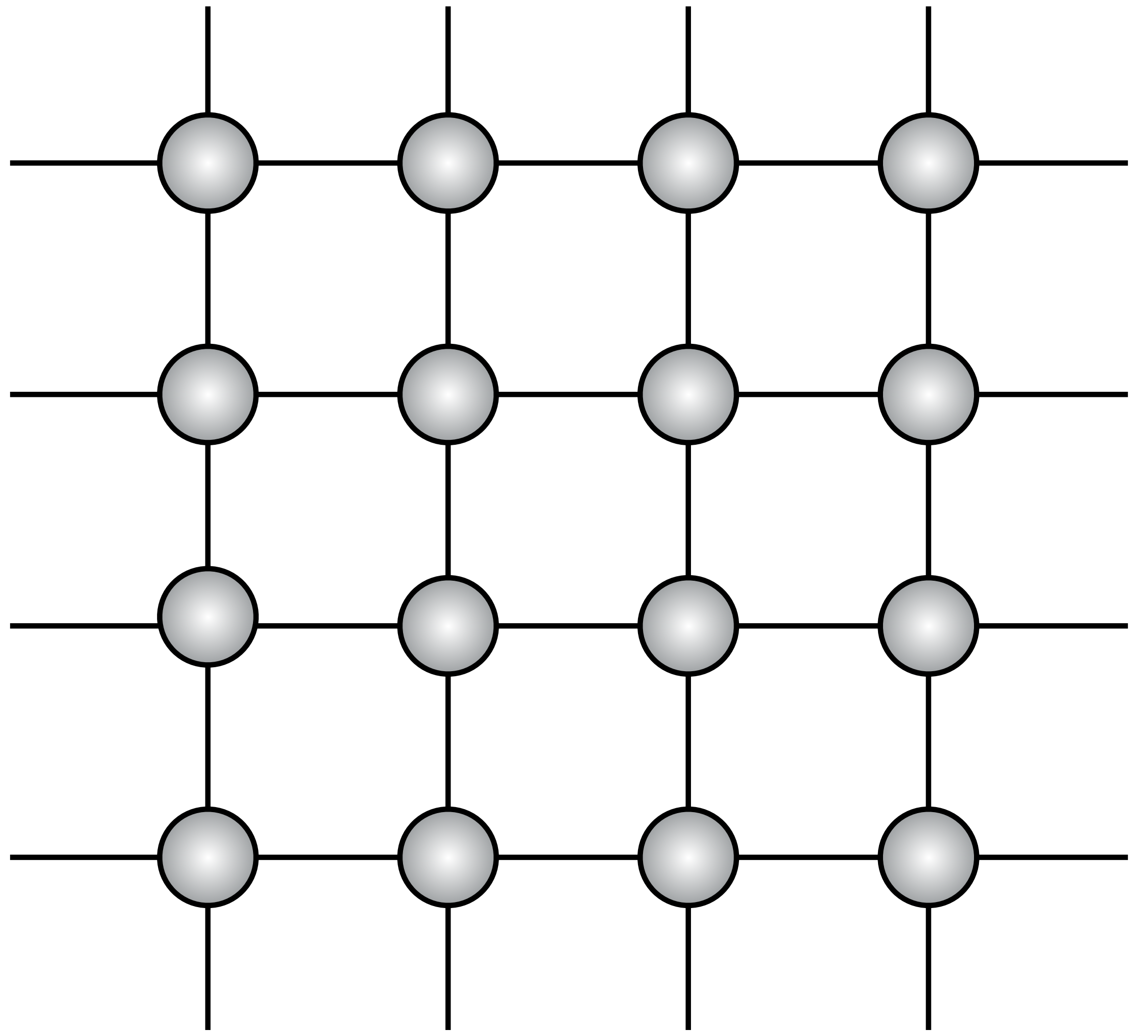


This Talk:

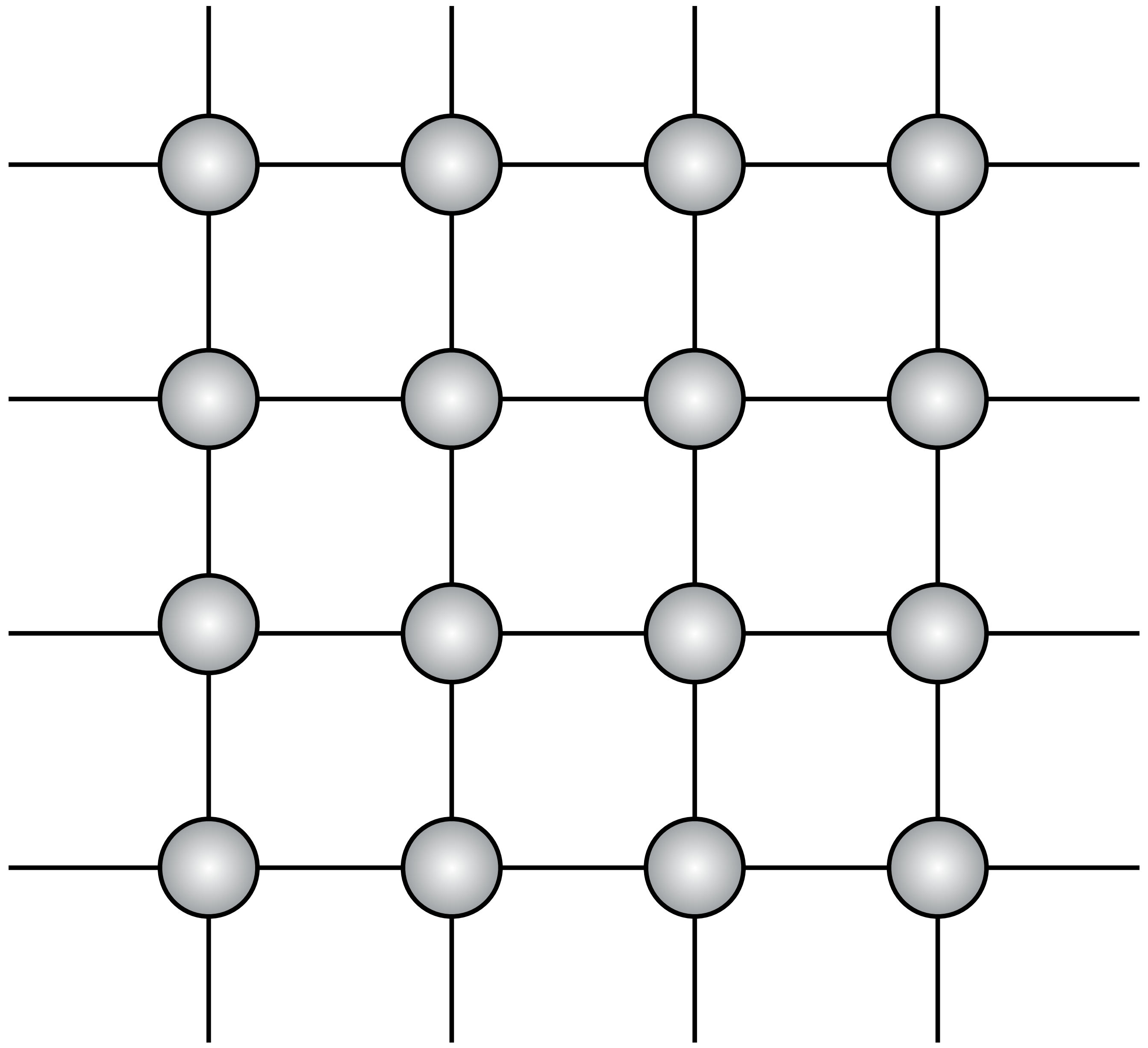
- How to extract S-matrix elements from Lattice QCD:
  - Two-particle scattering in finite volume
  - Lattice results for three particles

Lattice QCD basics



Lattice QCD Basics (I)

- Lattice QCD is the state-of-the-art treatment of the strong interaction at hadronic energies



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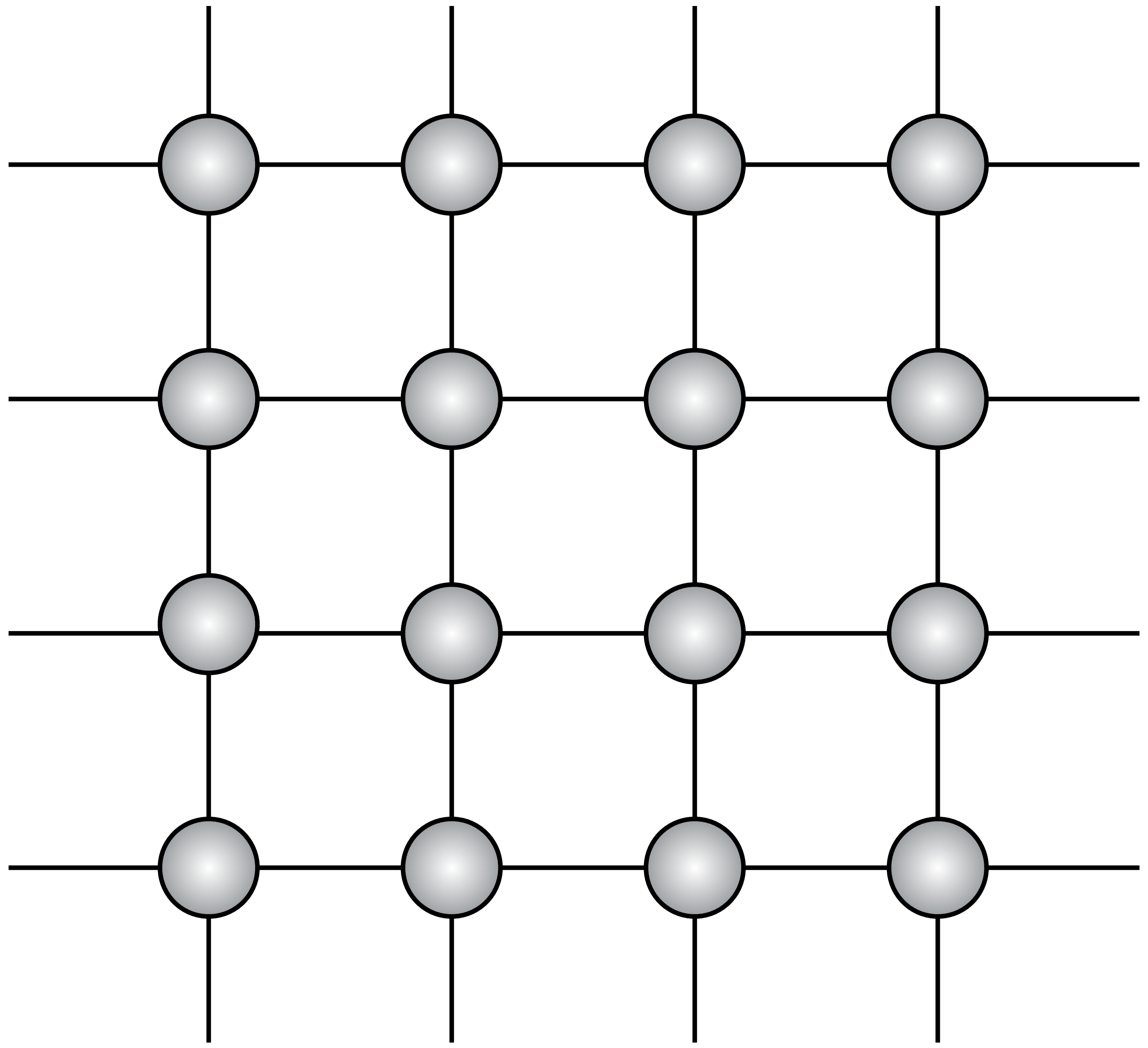
$$\mathcal{Z} = \int D\psi D\bar{\psi} DA e^{-S_E(\psi,\bar{\psi},A_\mu)}$$



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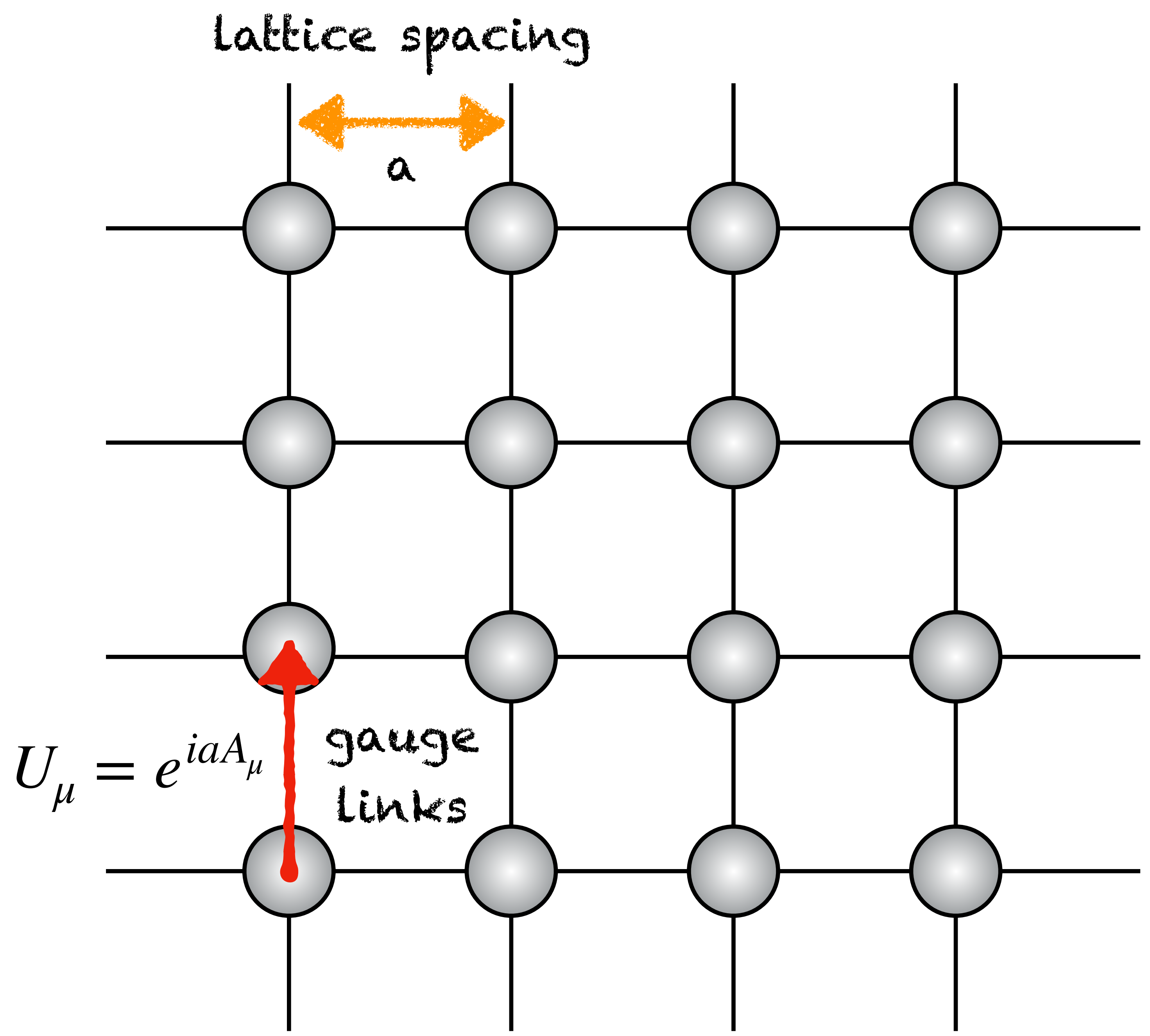
    $\rightarrow$Under control but technical

    (e.g., discretization effects and continuum limit)



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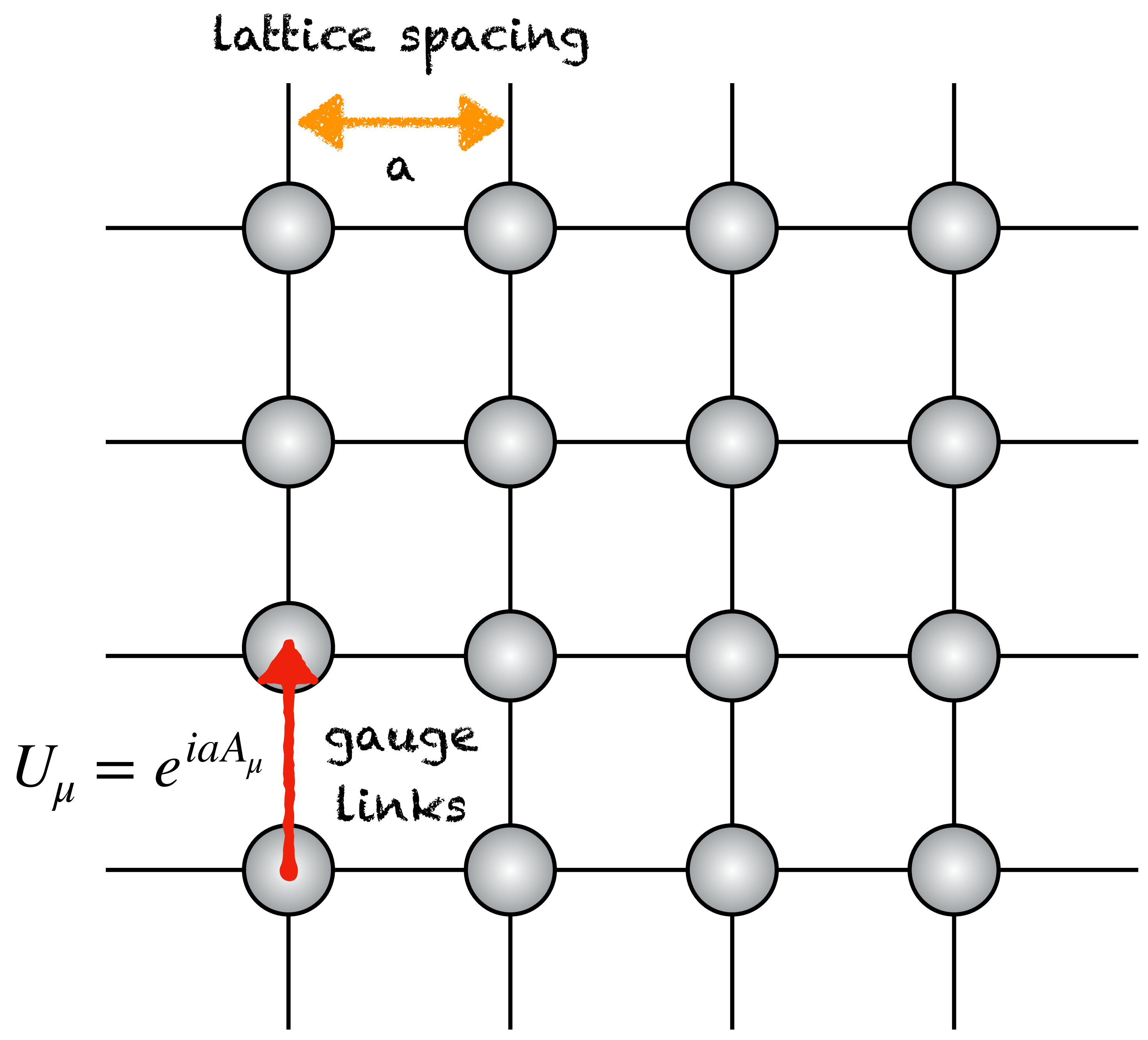
- Compute correlation functions

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{\mathcal{Z}}\int D\psi D\bar{\psi} DA\ \mathcal{O}(t)\mathcal{O}(0) e^{-S(\psi,\bar{\psi},A_\mu)}$$



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- In Lattice QCD, we measure energy levels and matrix elements: "Spectral decomposition"

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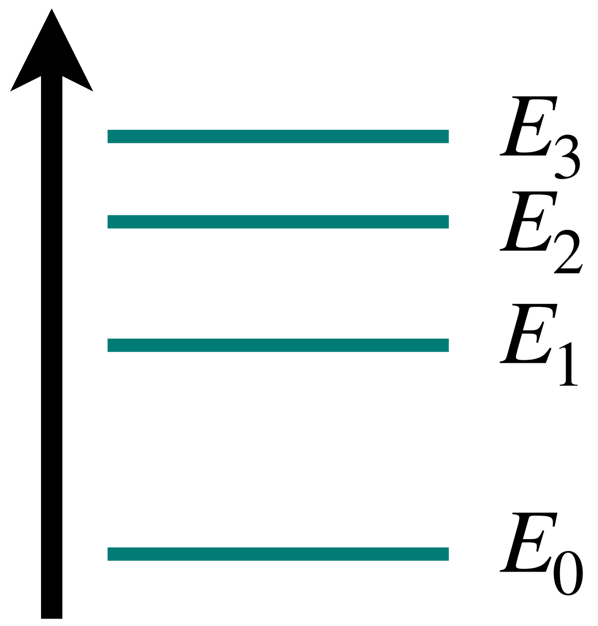
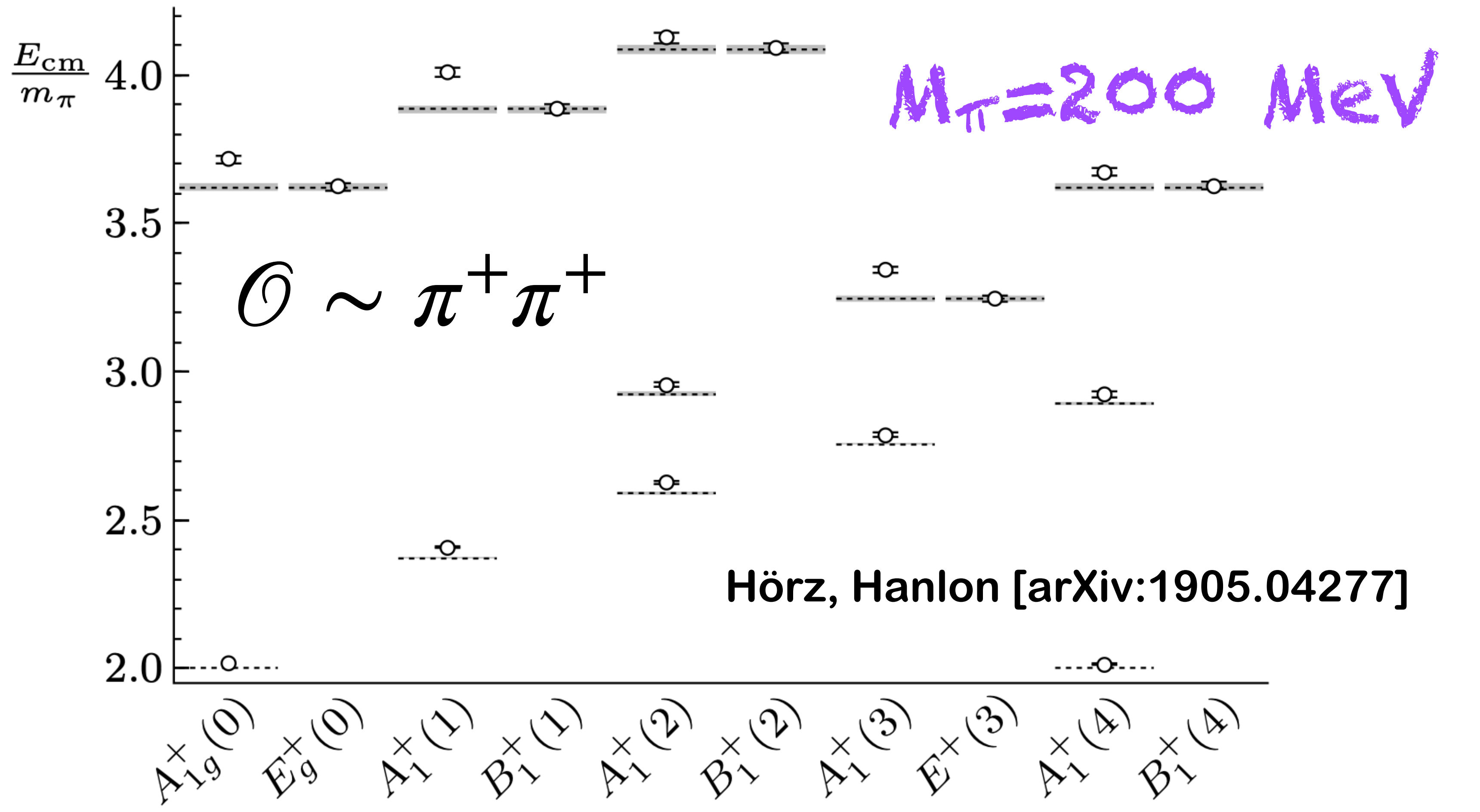
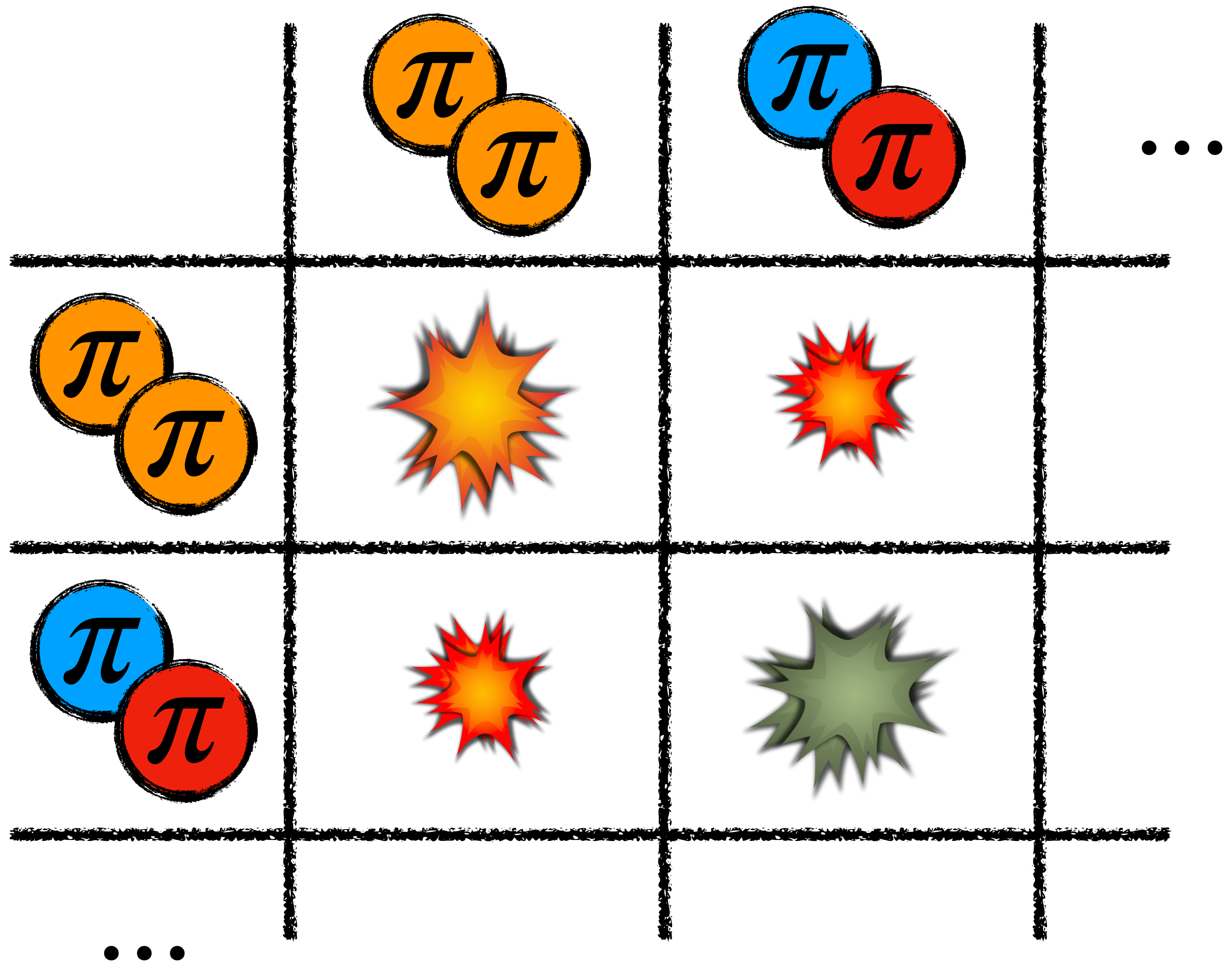
- **Multiple operators to obtain several energy levels**

The Spectrum





Two particles in finite-volume



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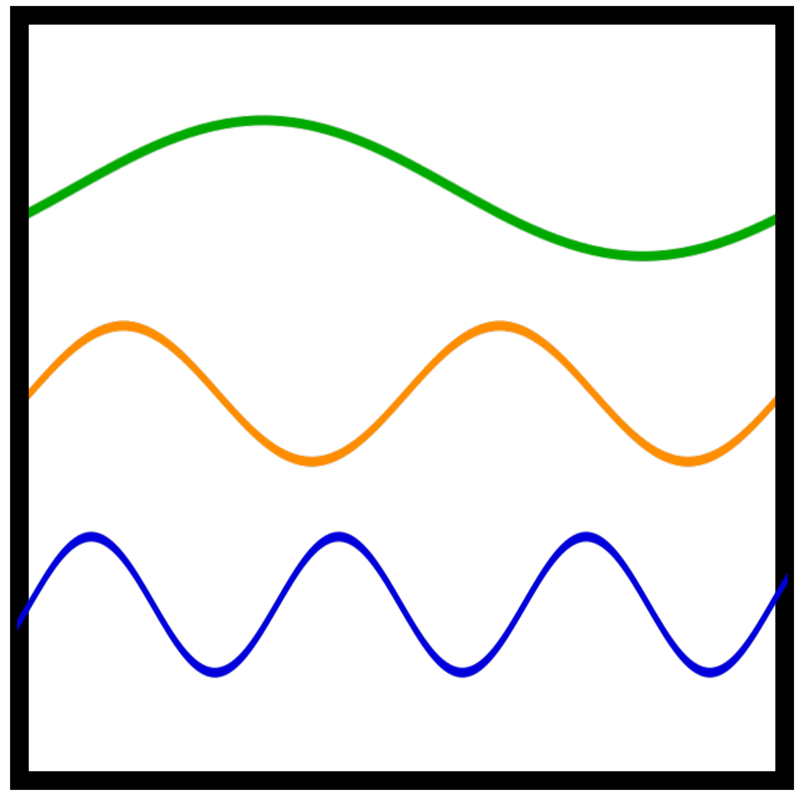
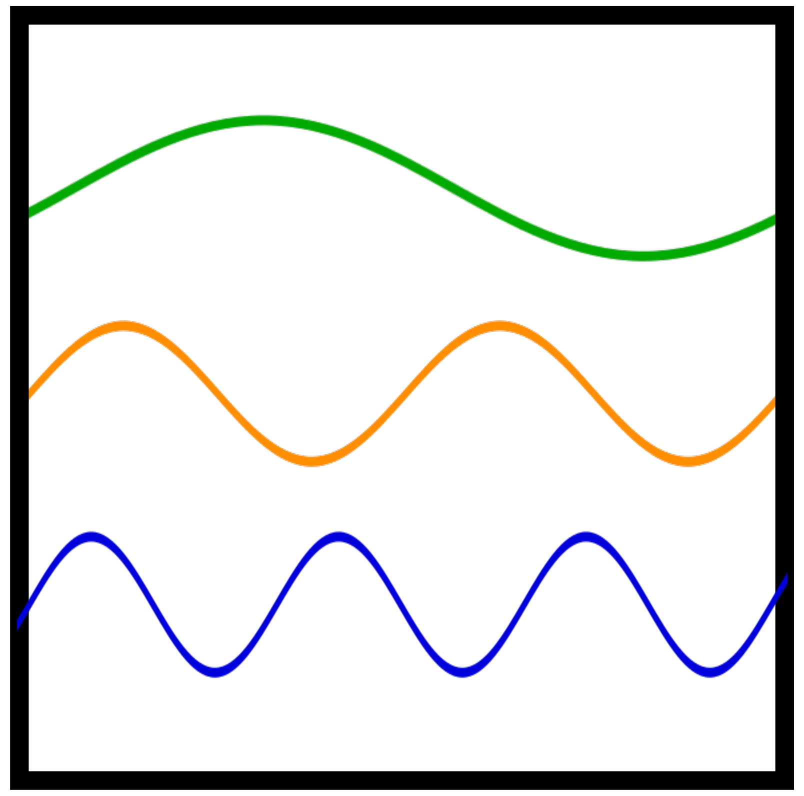


Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC

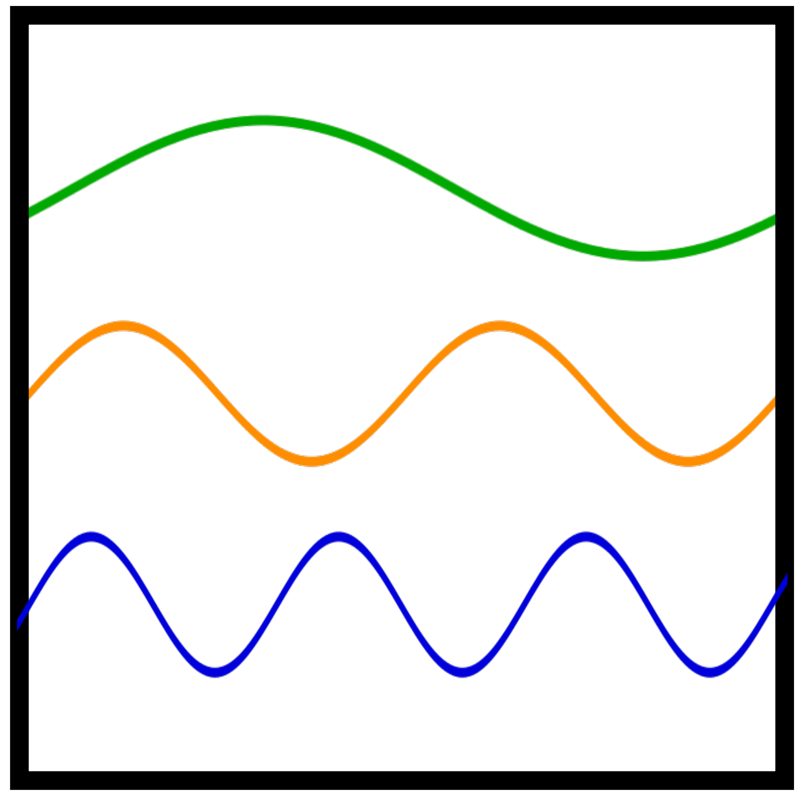


$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^{\,2}}$

Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC → Interactions change the spectrum: it can be treated as a perturbation



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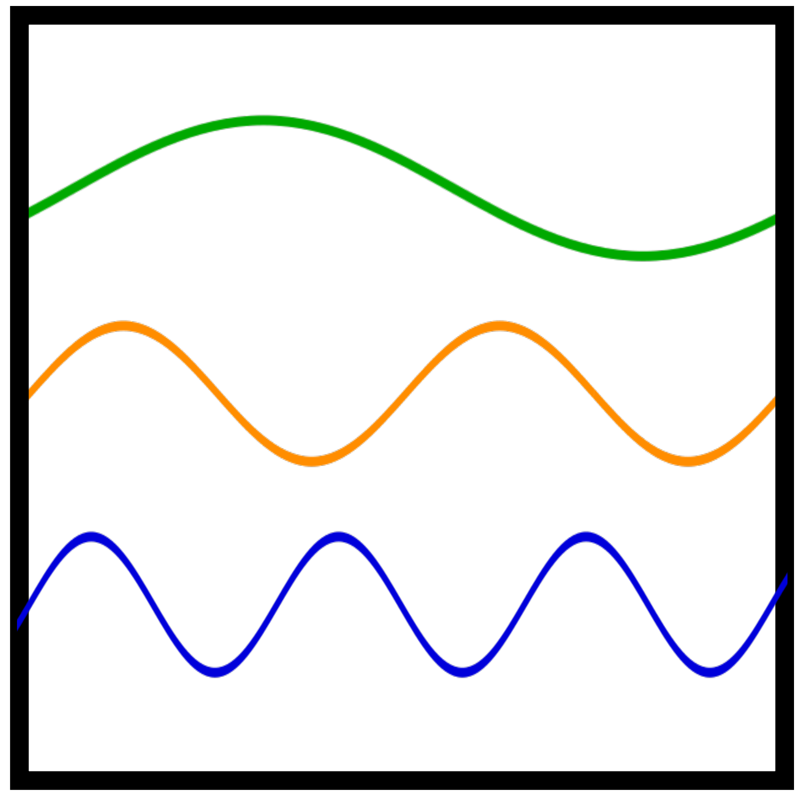

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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

$$\Delta E_2 = \frac{\mathcal{M}_2(E = 2m)}{8m^2L^3} + O(L^{-4})$$

[Huang, Yang, 1958]

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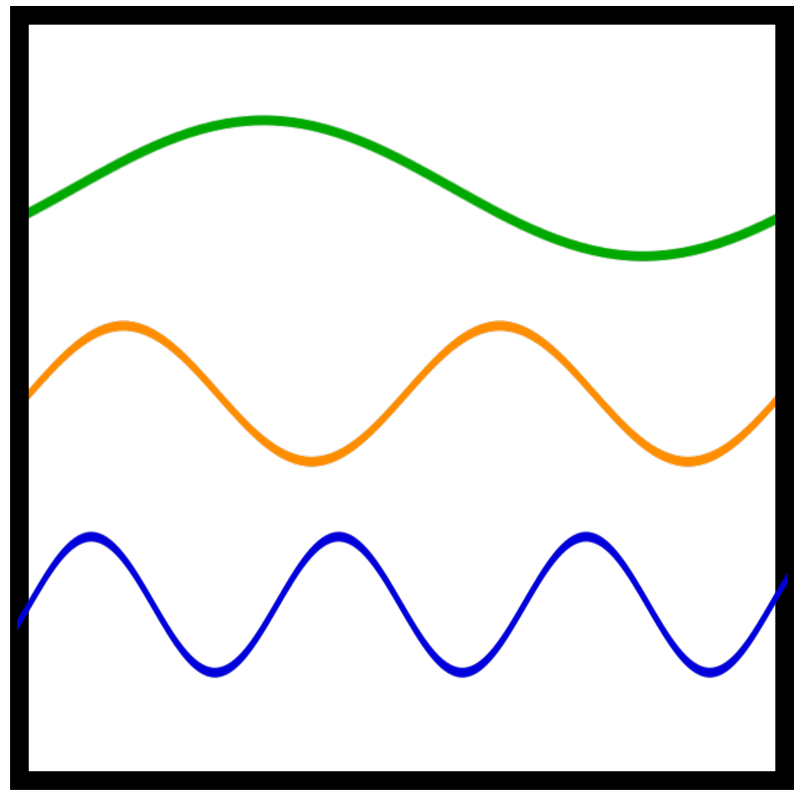

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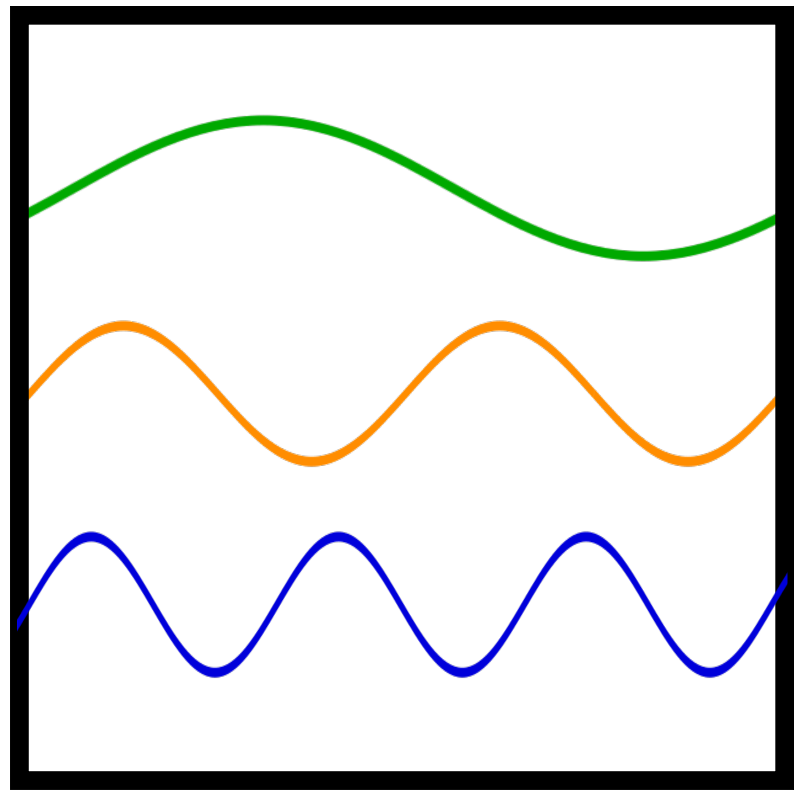

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In general a problem of
Quantum Field Theory
in finite volume

Interactions change the spectrum:
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The Lüscher Formalism

- A new field was opened by M. Lüscher in ‘86

finite-volume spectrum of two identical scalars → s-wave scattering amplitude

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

II. Scattering States

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Theory Division, Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Federal Republic of Germany

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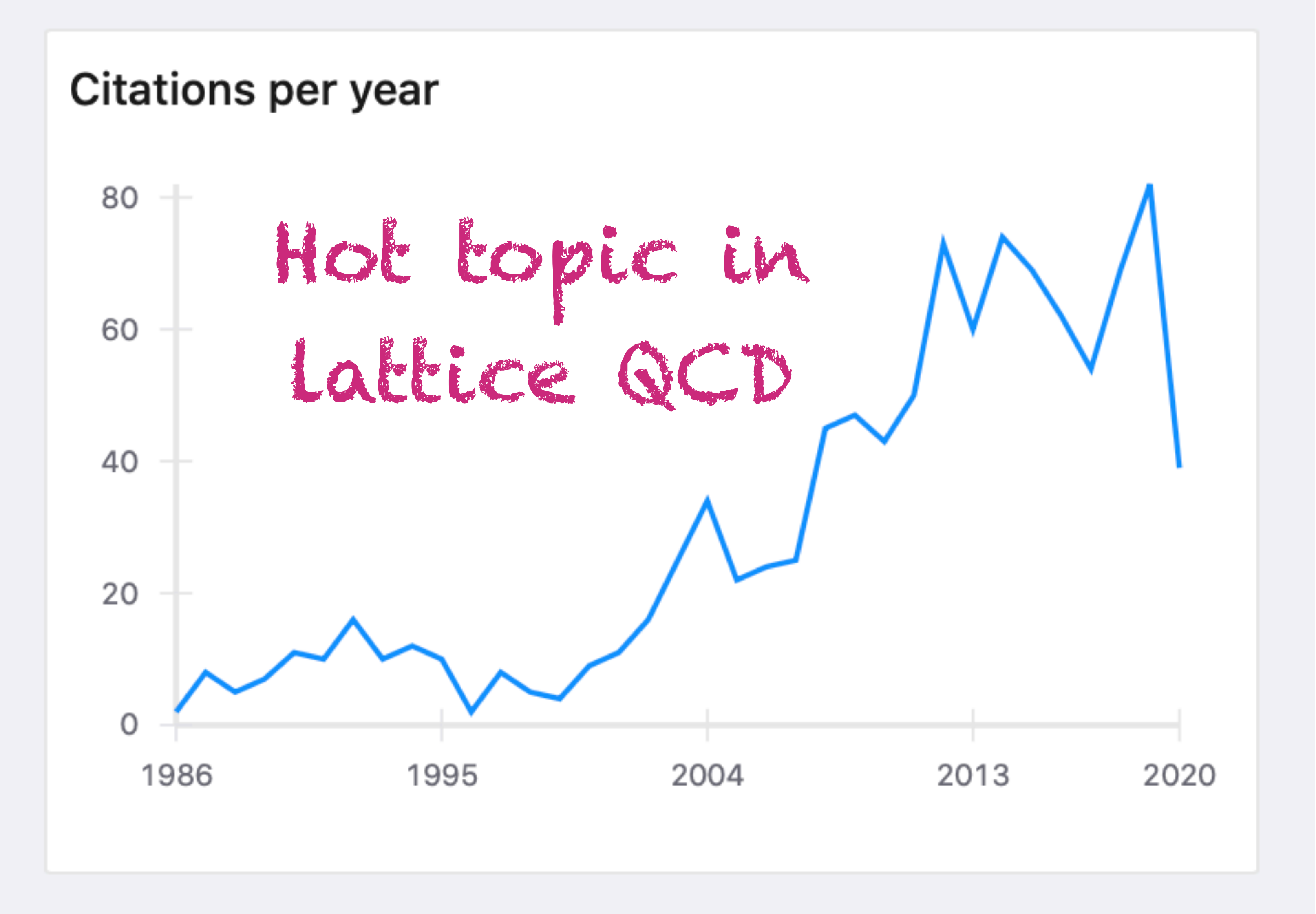

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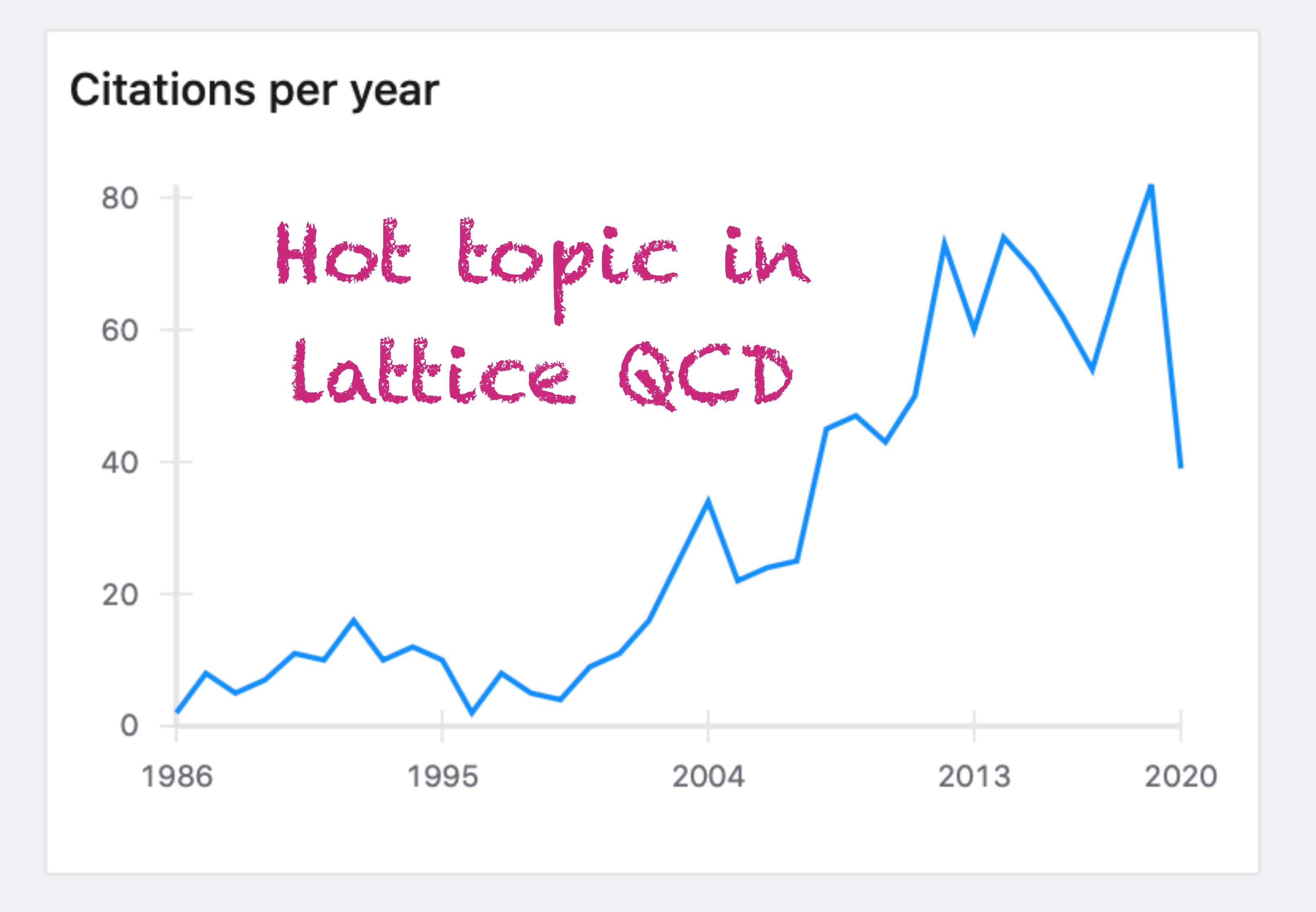

- Fully general formalism exists up to date:
  - Multichannel, non-identical $2 \to 2$ scattering for particles with spin in all partial waves. Including for weak decays, such as $K \to \pi\pi$ (Lellouch-Lüscher)
  - Many people have contributed over the years:
    - Rummukainen and Gottlieb
    - Kim, Sachrajda and Sharpe
    - Göckeler, Horsley, Lage, Meißner, Rakow, Rusetsky, Schierholz, Zanotti
    - Briceño

Quantization Condition(I)

- In order to derive the full relation, consider the finite-volume correlator:

$$C_L(E, \overrightarrow{P}) = \int e^{iPx} \langle \mathcal{O}(x) \,|\, \mathcal{O}(0) \rangle =$$

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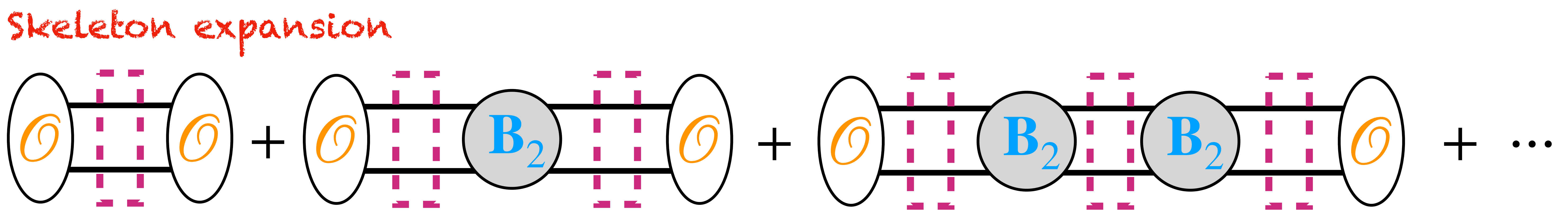

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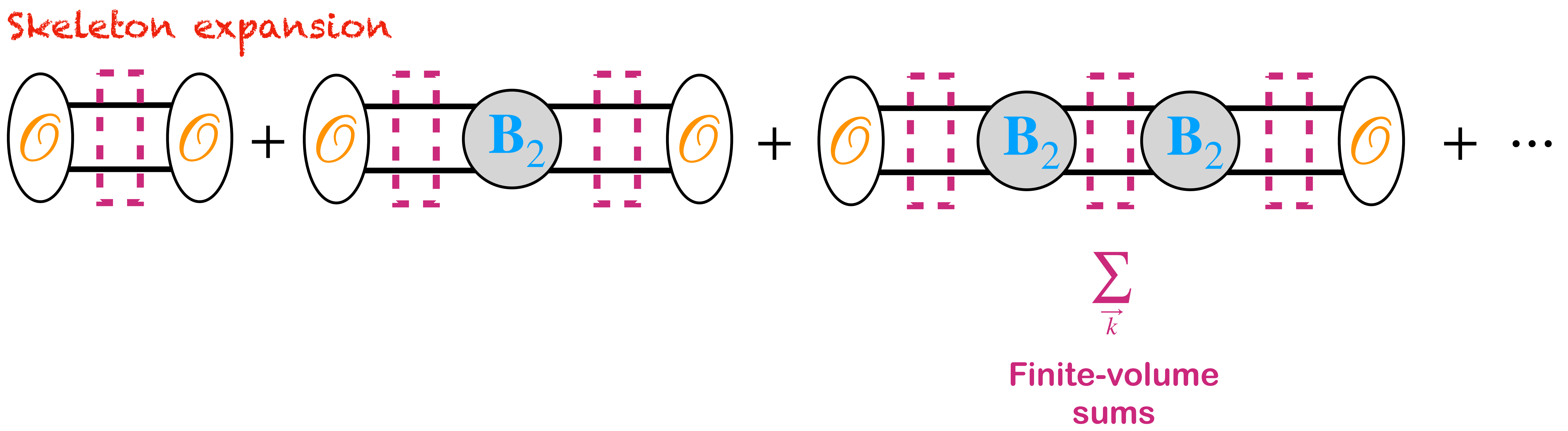

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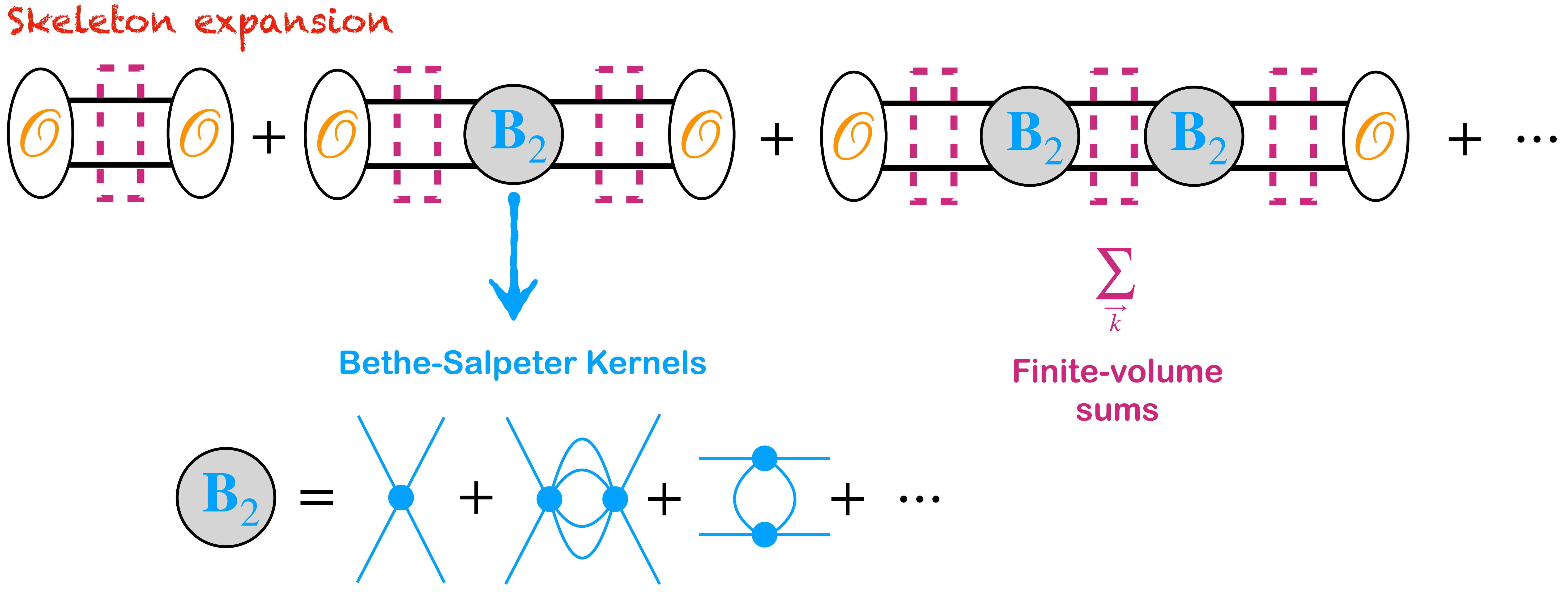

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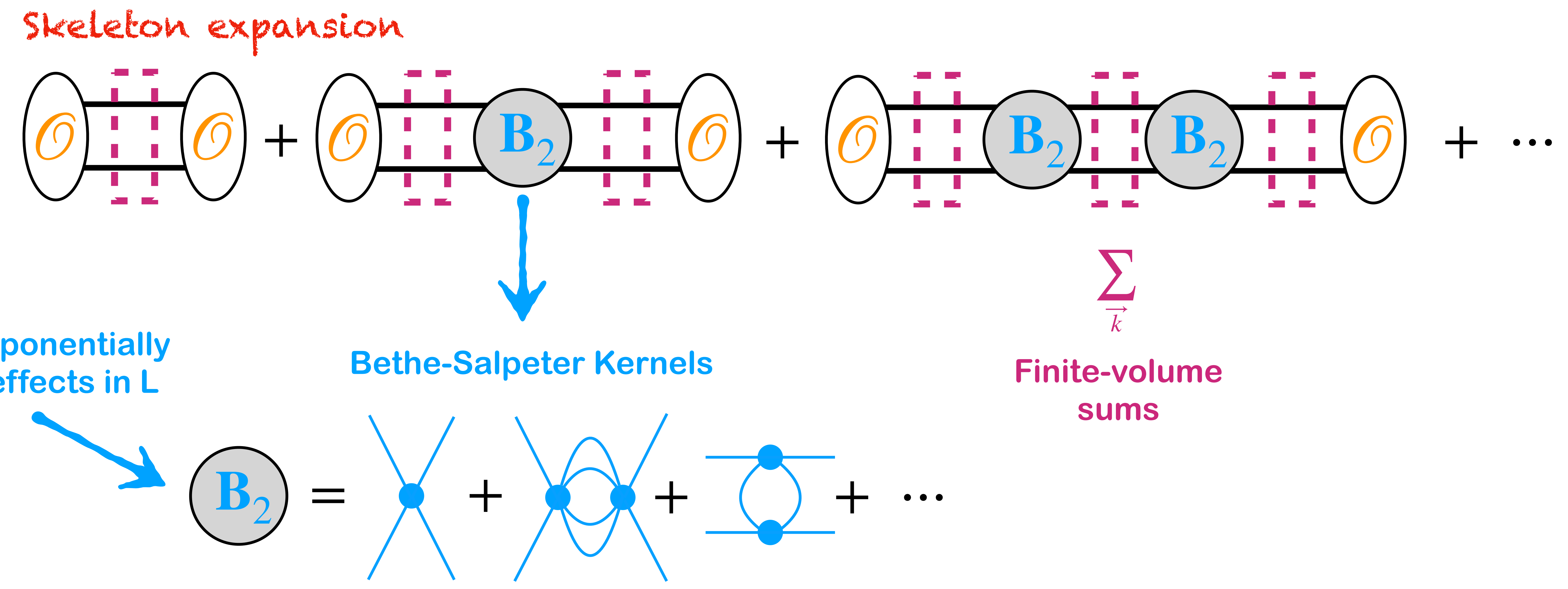

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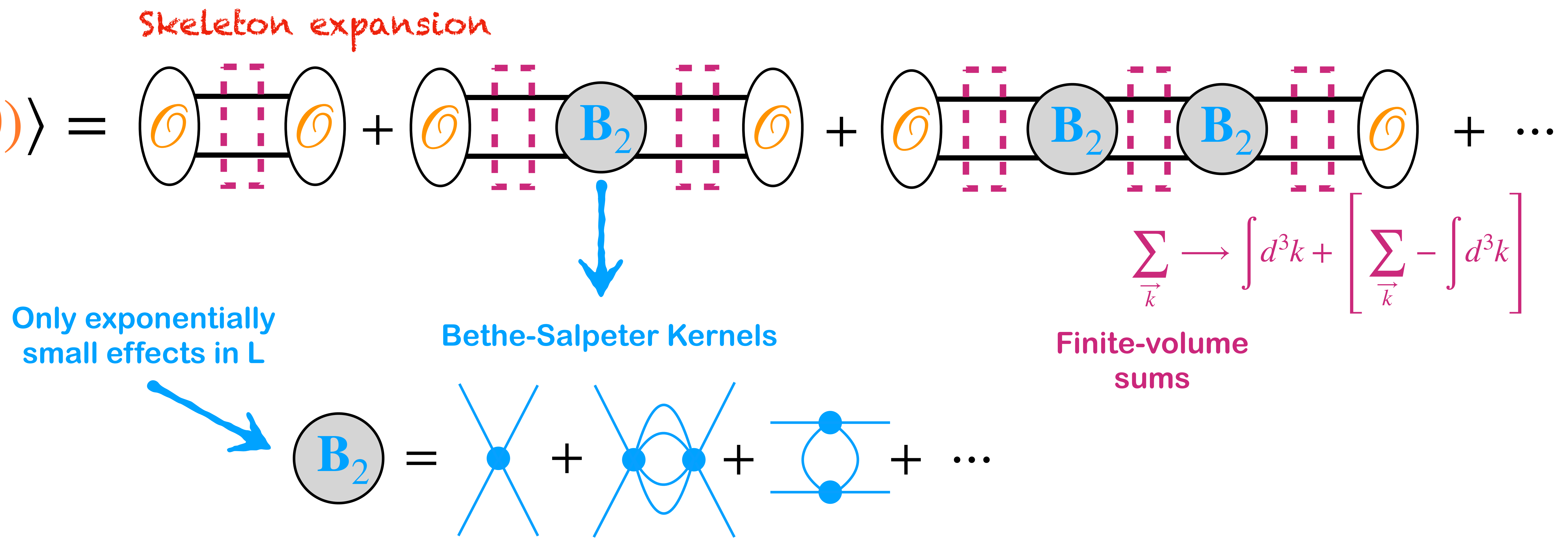

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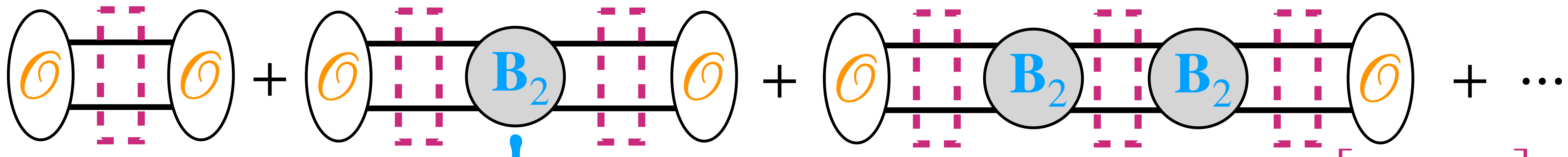


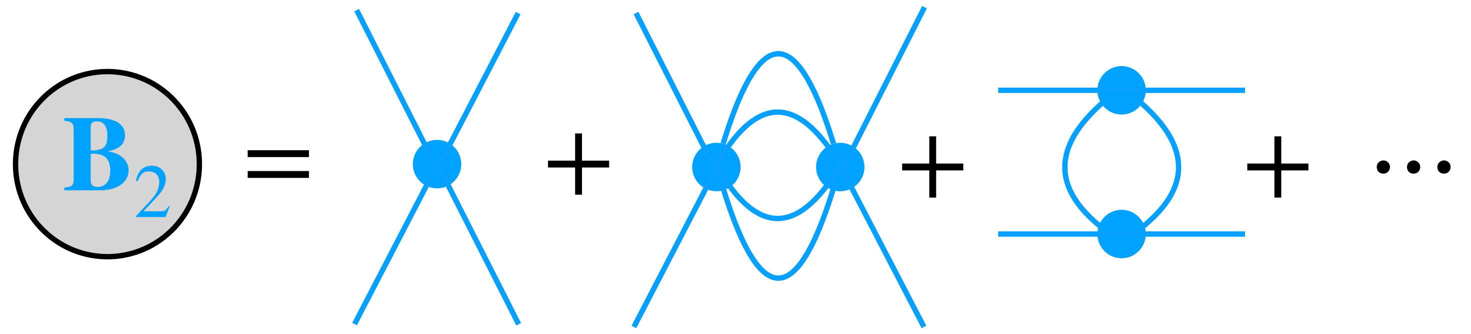
[à la Kim, Sachrajda, Sharpe]

$$\sum_{\vec{k}} \longrightarrow \int d^3k + \left[\sum_{\vec{k}} - \int d^3k \right]$$

Finite-volume sums

Bethe-Salpeter Kernels

Only exponentially small effects in L



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2. Resumation of diagrams

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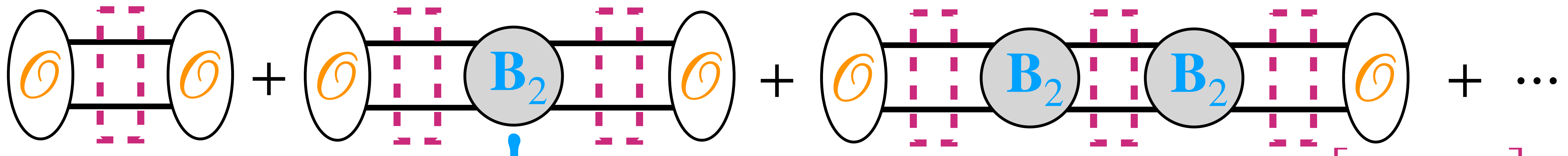

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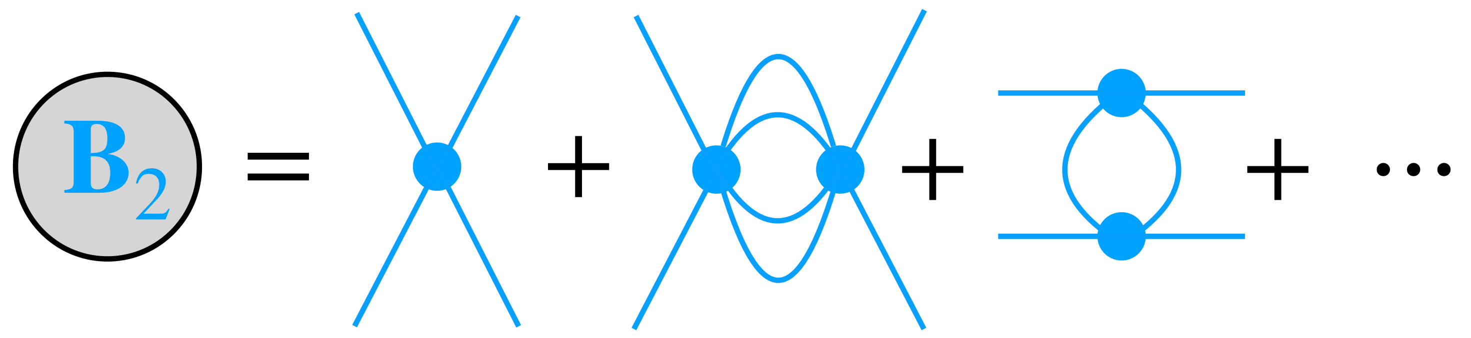

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$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A^\dagger \frac{1}{\mathcal{K}_2 + F^{-1}} A + O(e^{-mL})$$

Known kinematic function

$$\mathcal{M}_2^{-1} = \mathcal{K}_2^{-1} - i\sqrt{s - 2m^2}$$

Quantization Condition(II)

$$C_L(E, \overrightarrow{P}) = \quad \textbf{some algebra ...} \quad = C_\infty(E, \overrightarrow{P}) + A^\dagger \frac{1}{\mathscr{K}_2 + F^{-1}} A + O(e^{-mL})$$

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K-matrix parametrized in terms of phase shift

$$\mathscr{K}_2^\ell = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

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! It holds below $E_{cm} < 4m$

Two-particle Quantization Condition

$$\det\left[\mathcal{K}_2(E_n) + F^{-1}(E_n,\vec{P},L)\right] = 0$$

Scattering K-Matrix

Known kinematic function

"QC2"

Isospin-2 ππ scattering

Two pions in s-wave

$$\mathcal{K}_2^{s-wave}(E_n) = \frac{-1}{F_{00}(E_n, \overrightarrow{P}, L)}$$

one energy level $\longrightarrow$ a phase shift point

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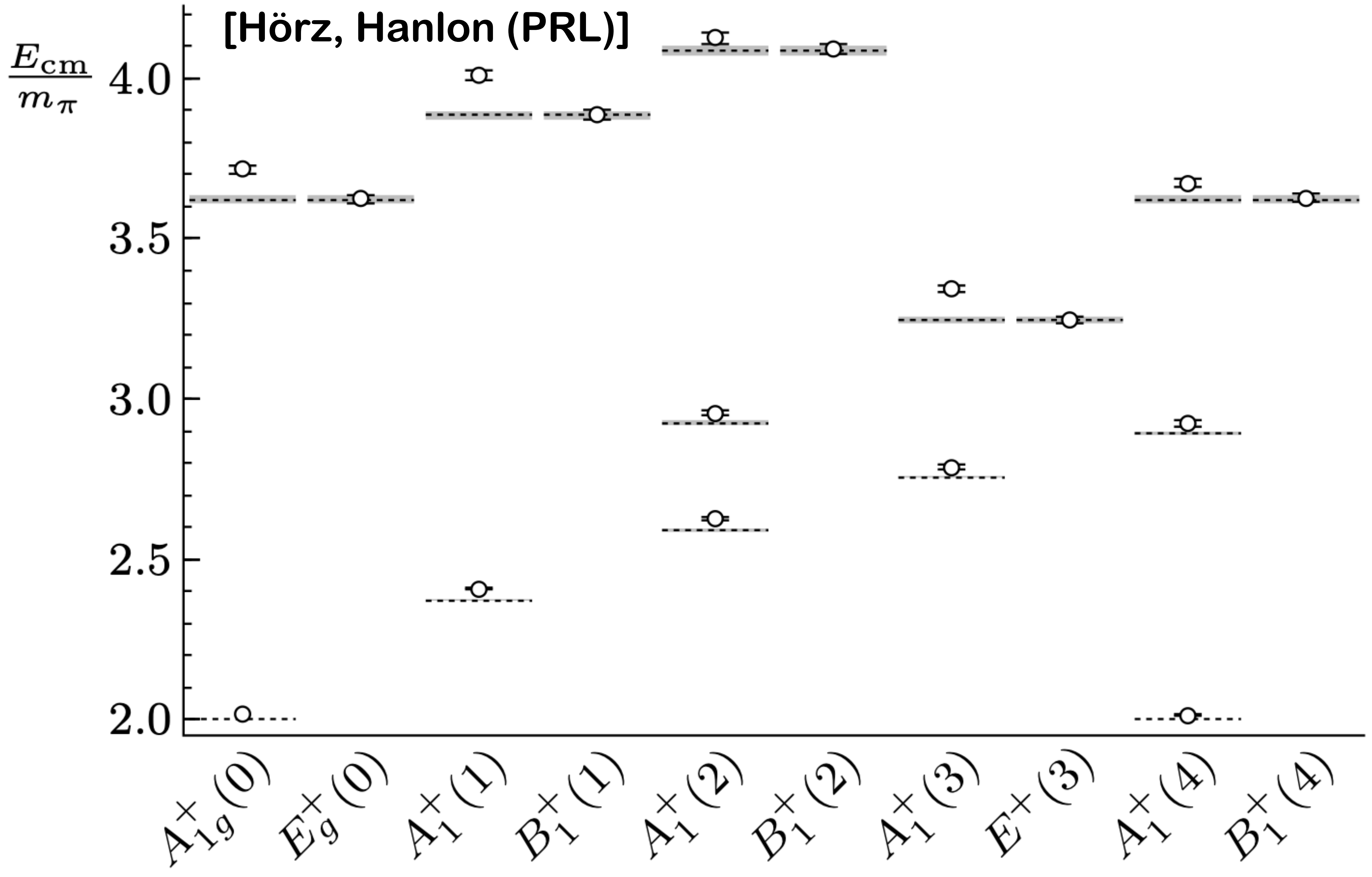

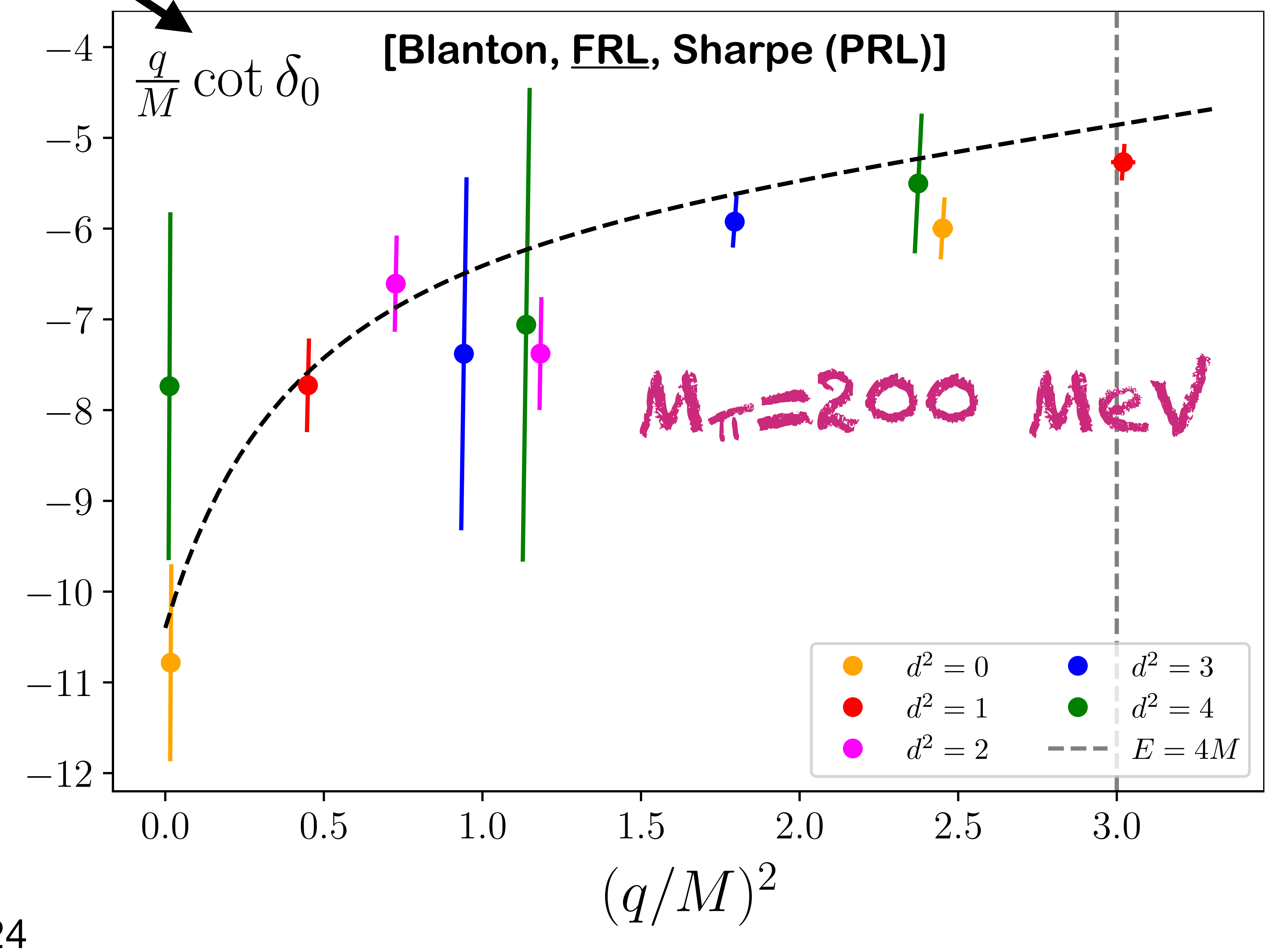

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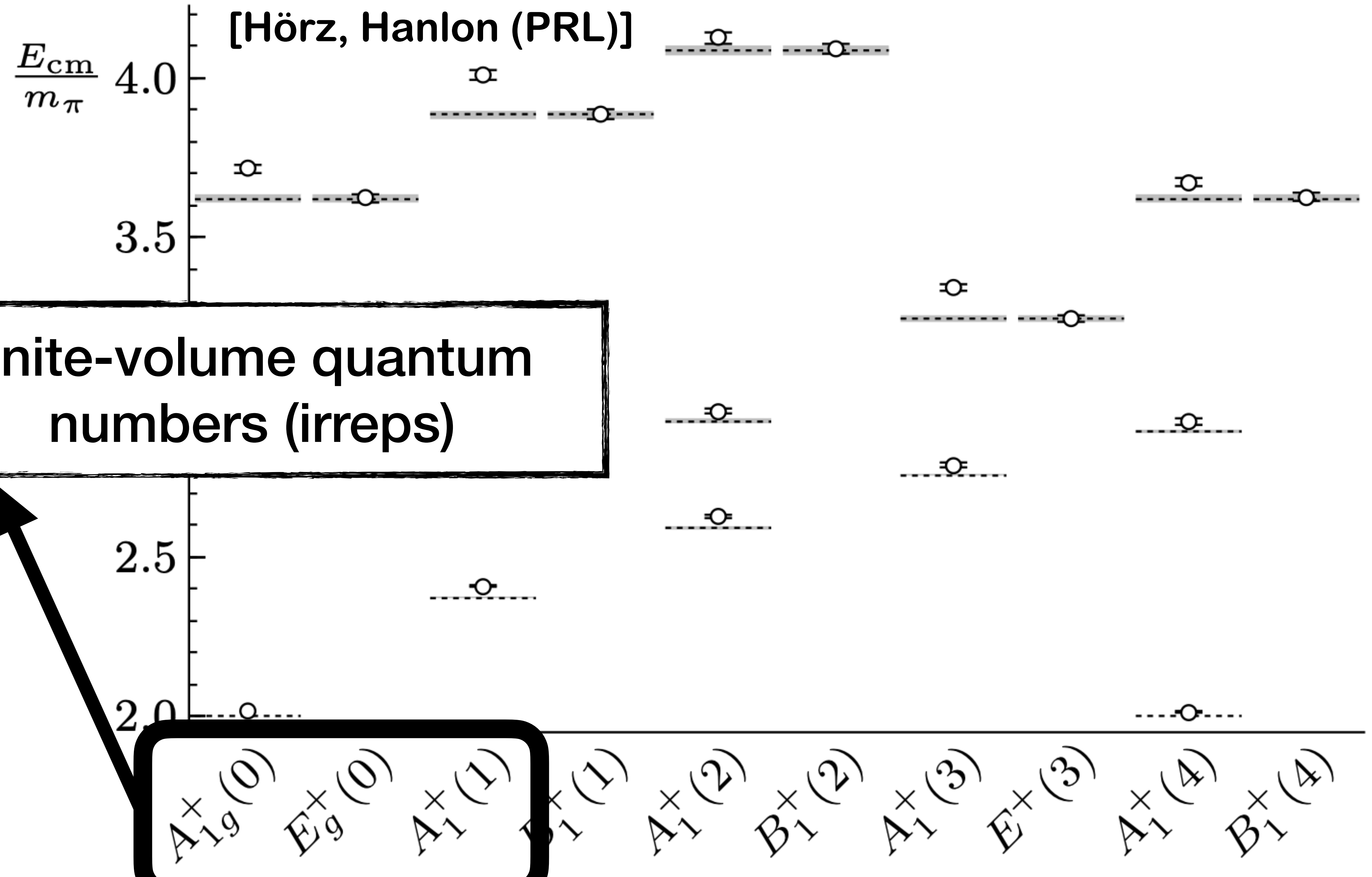
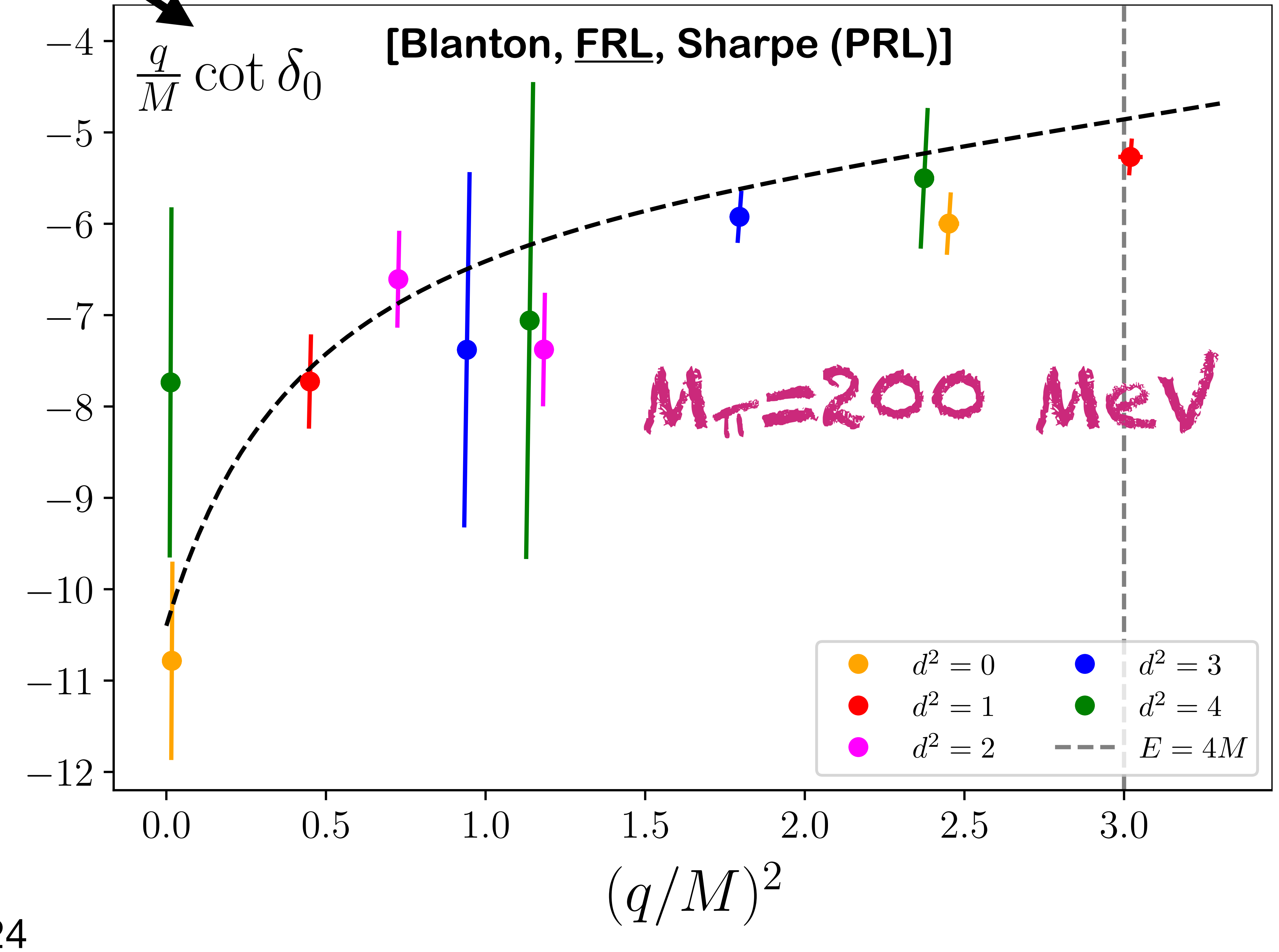
$$\mathscr{K}_2^{s-wave} \sim \frac{16\pi\sqrt{s}}{q\cot\delta_0}$$

one energy level → a phase shift point



finite-volume quantum numbers (irreps)



Isospin-2 ππ scattering

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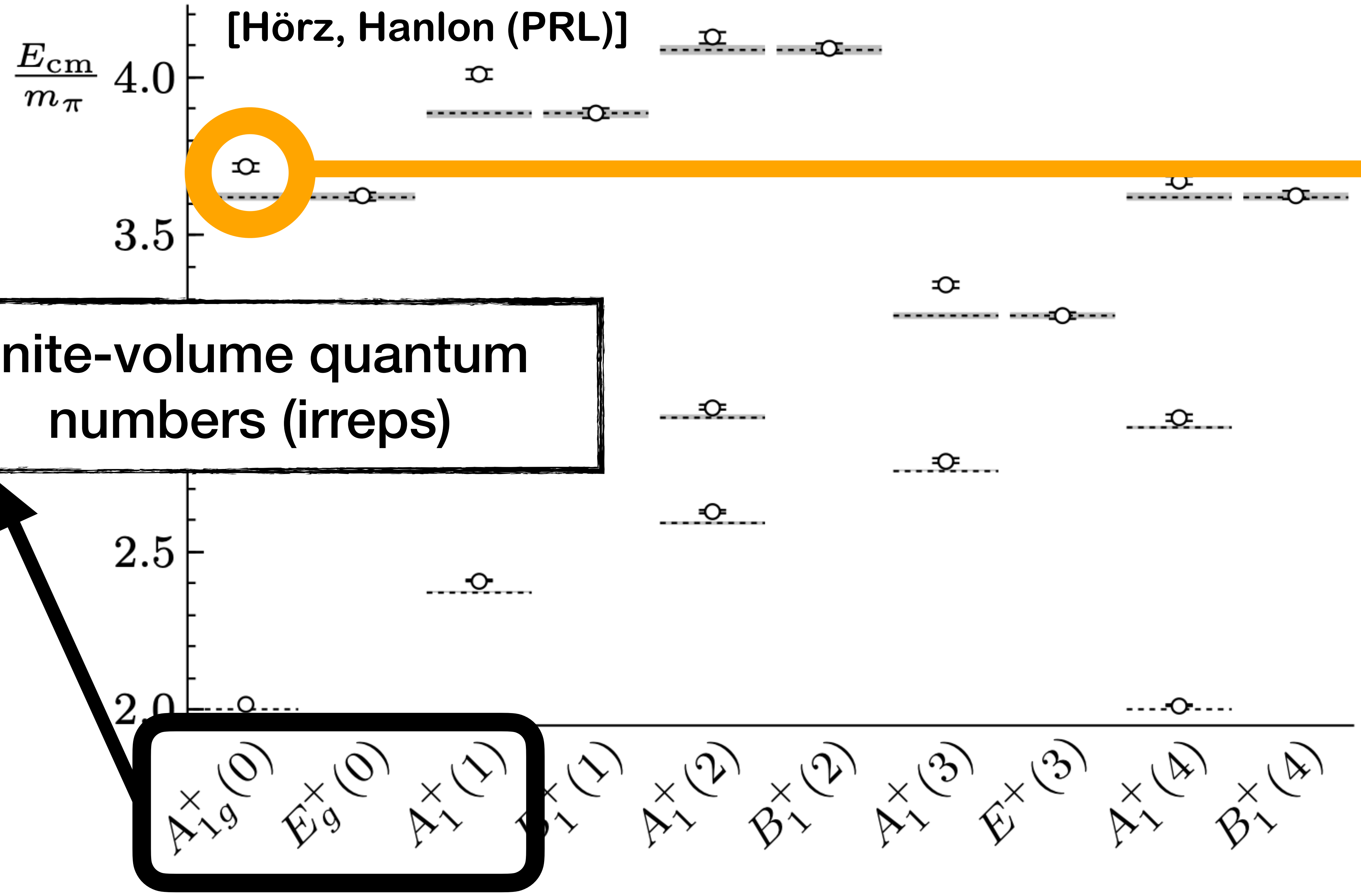

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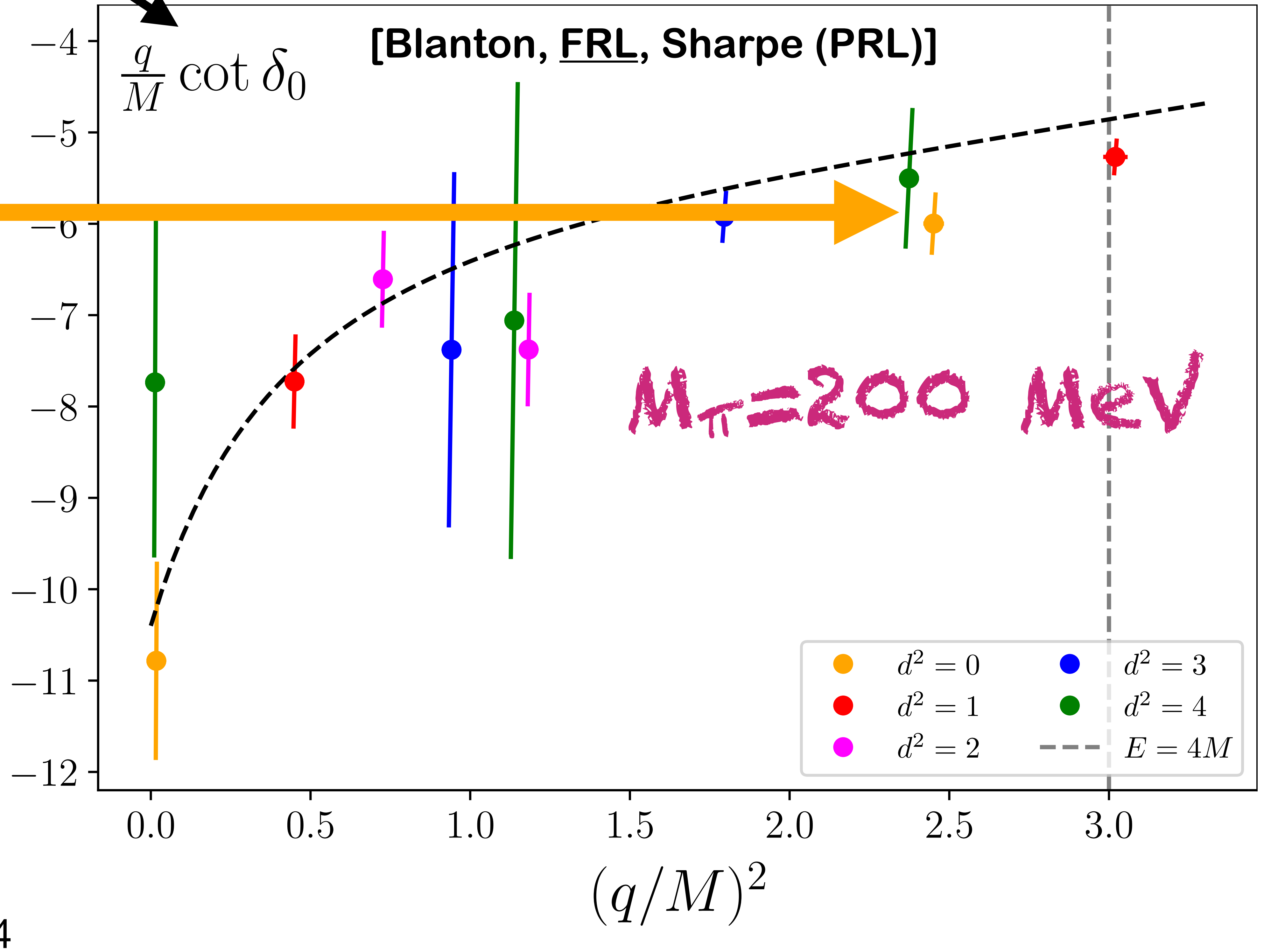

Isospin-2 $\pi\pi$ scattering

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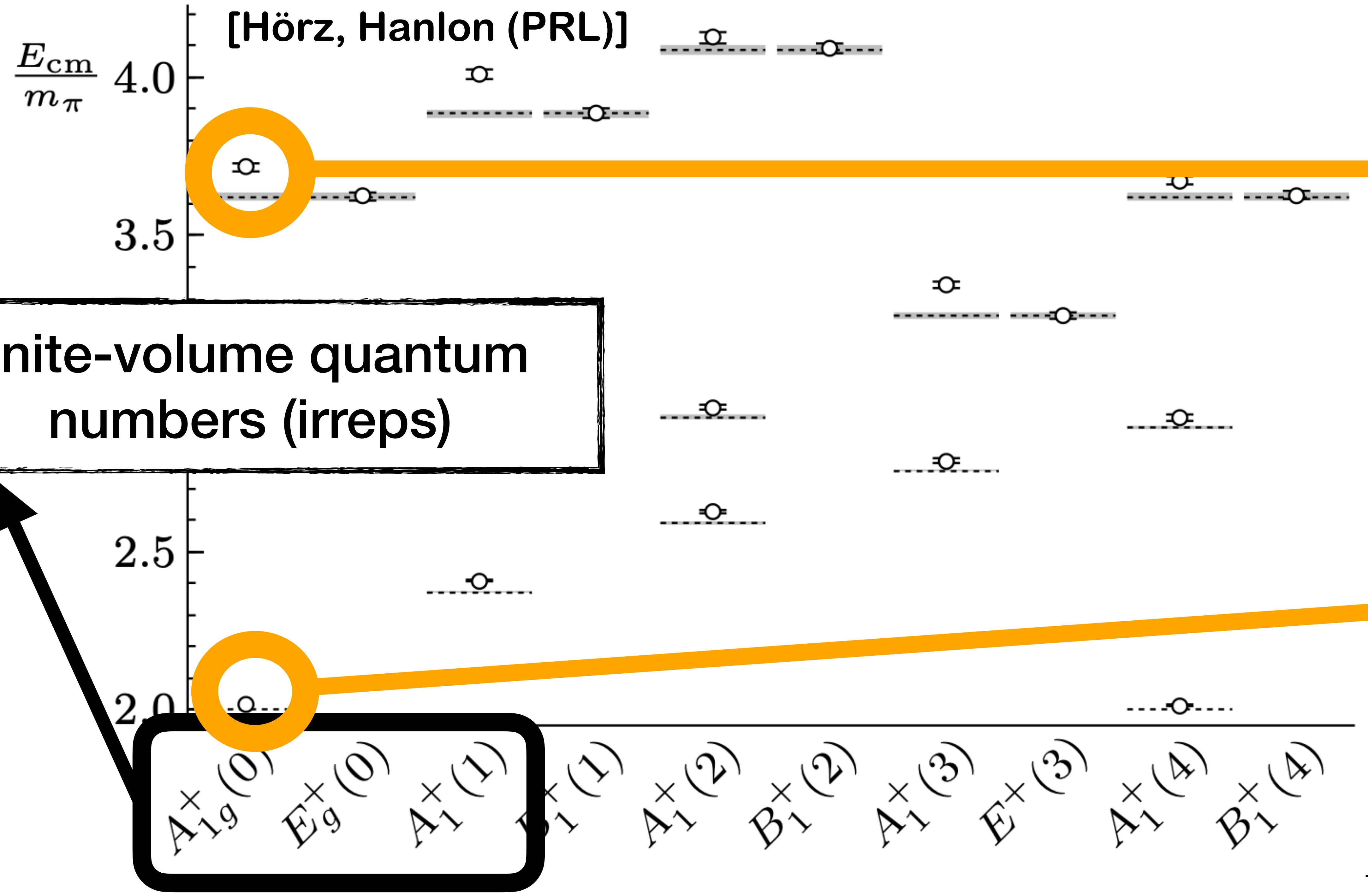

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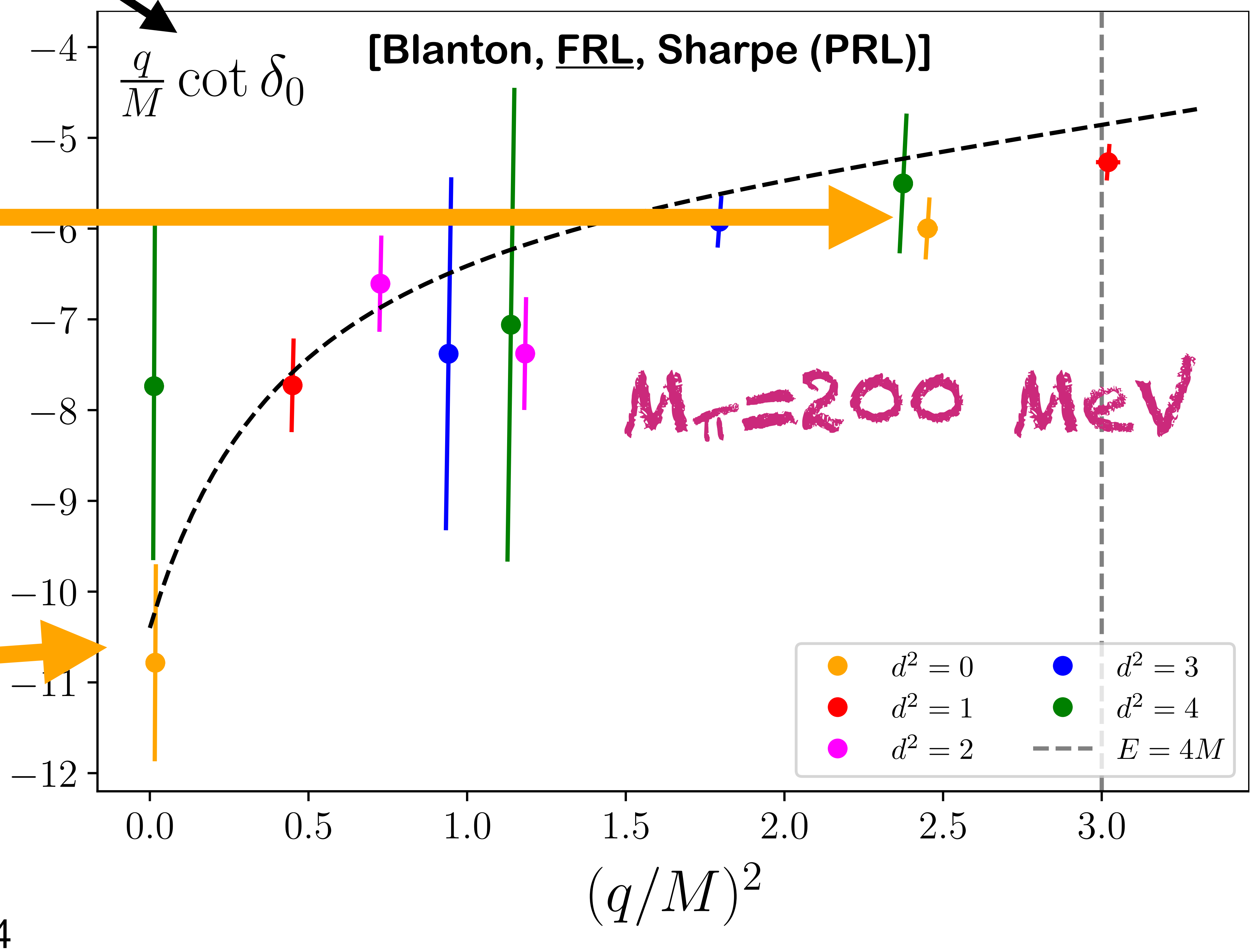

M_π=200 MeV

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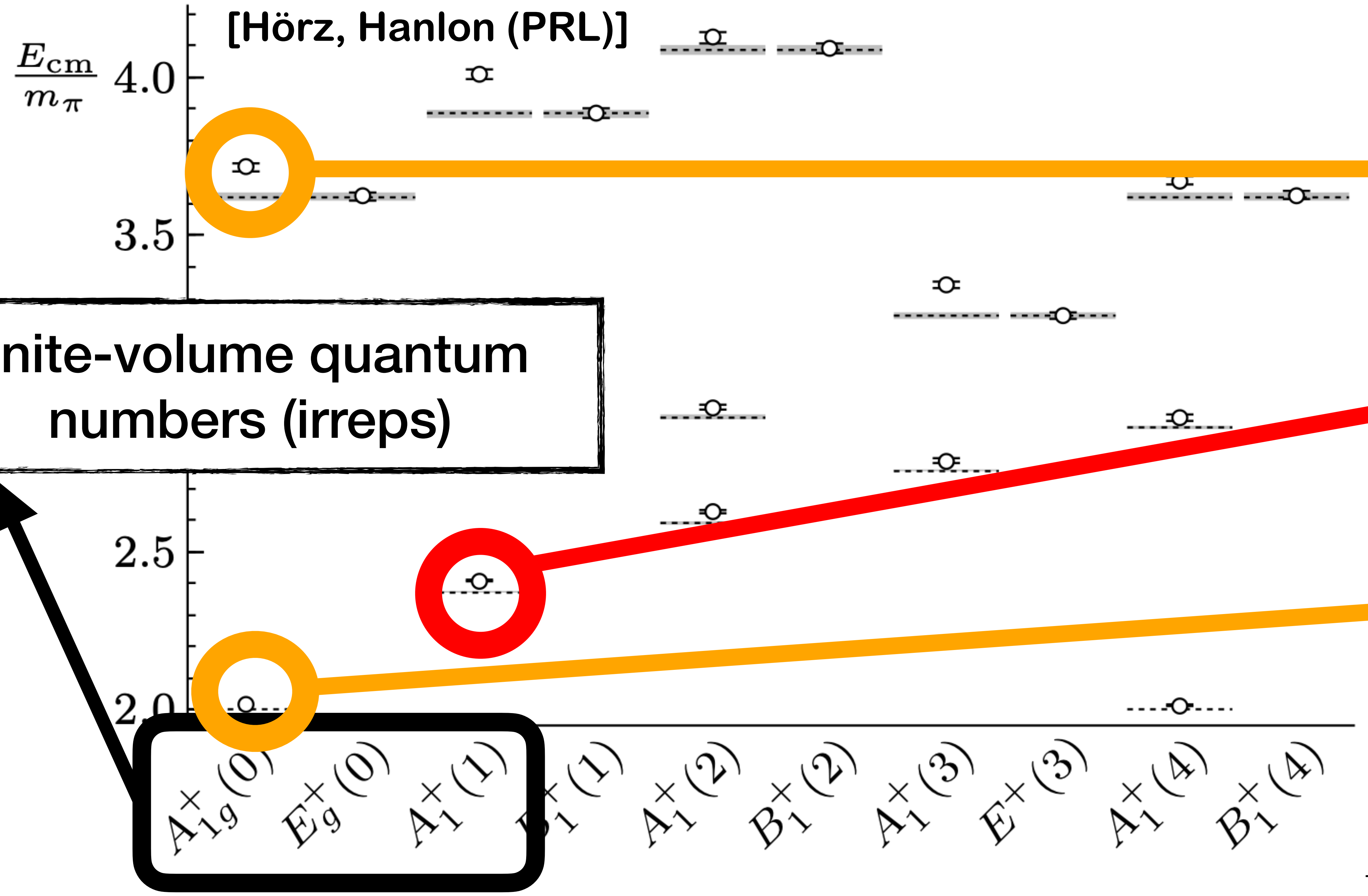

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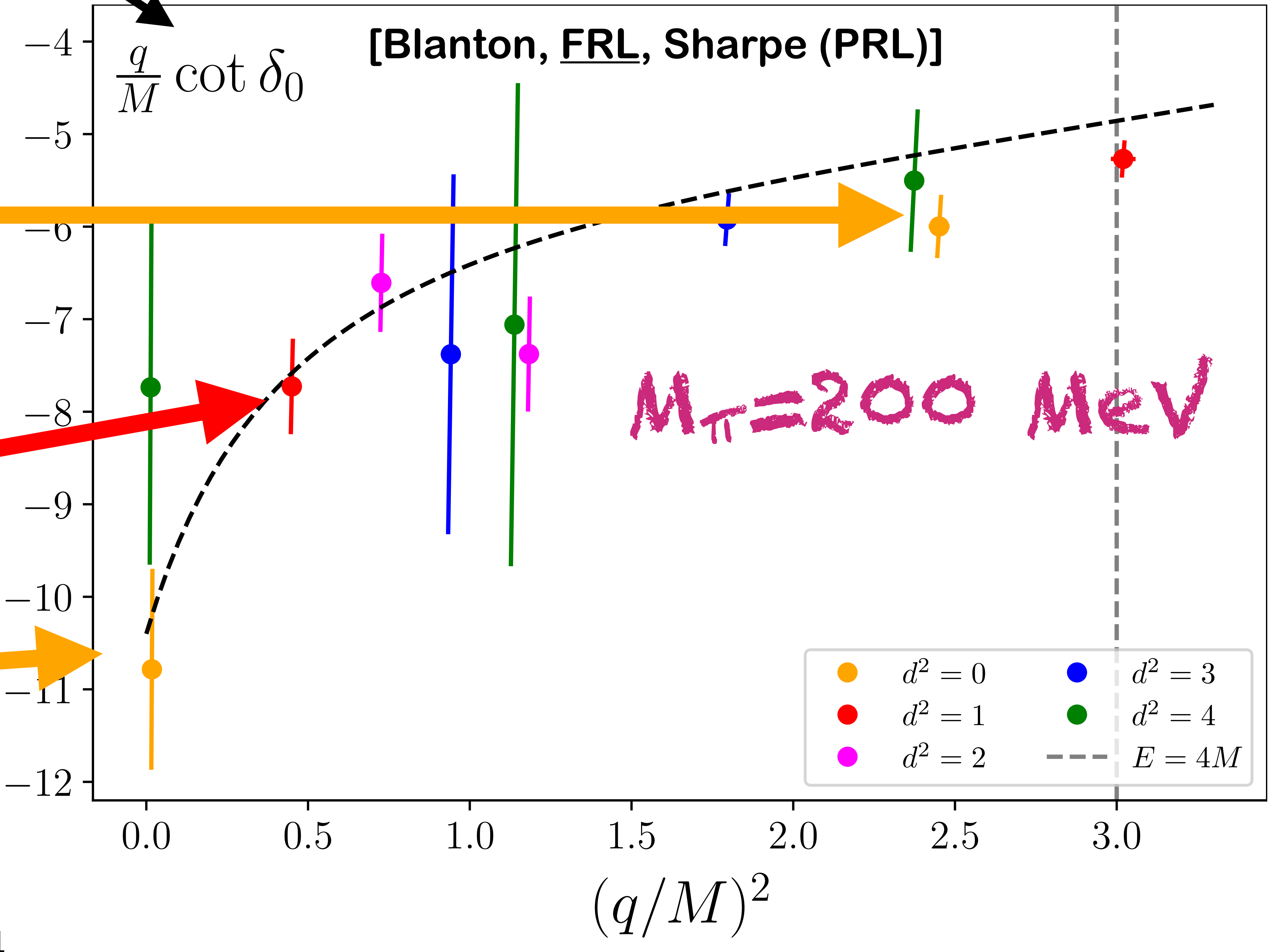

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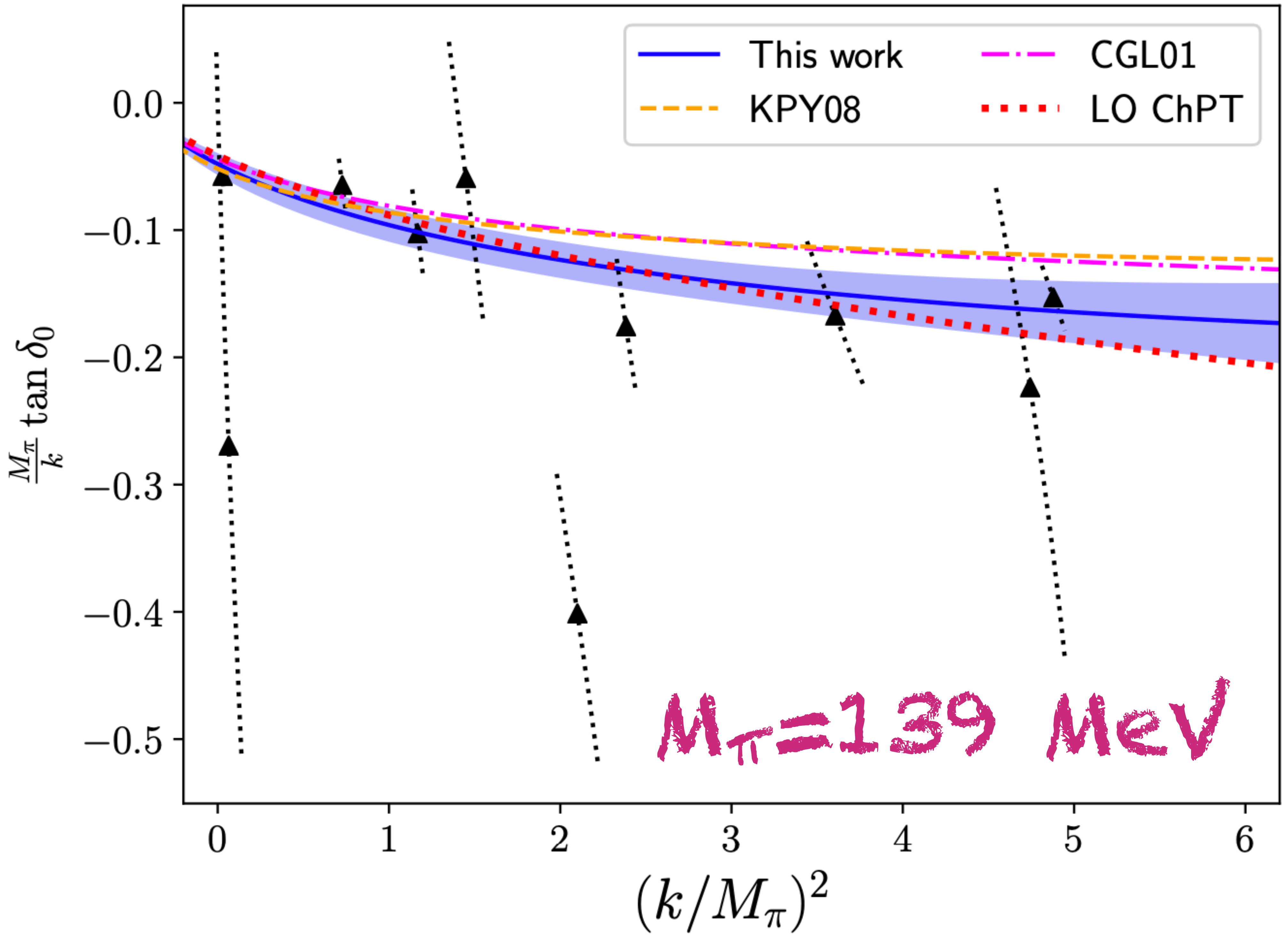

[Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC)]

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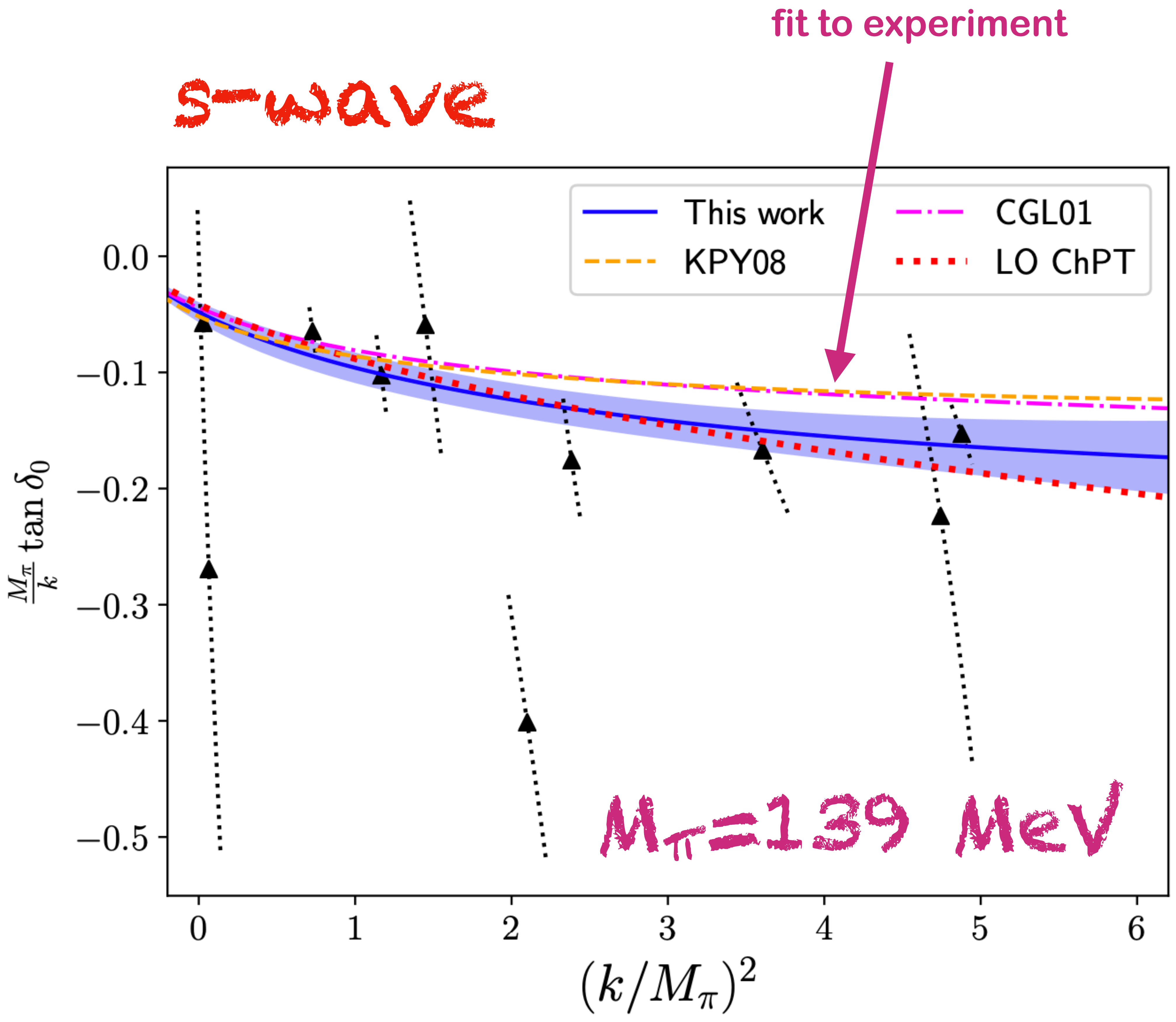

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Isospin-2 ππ scattering

fit to experiment

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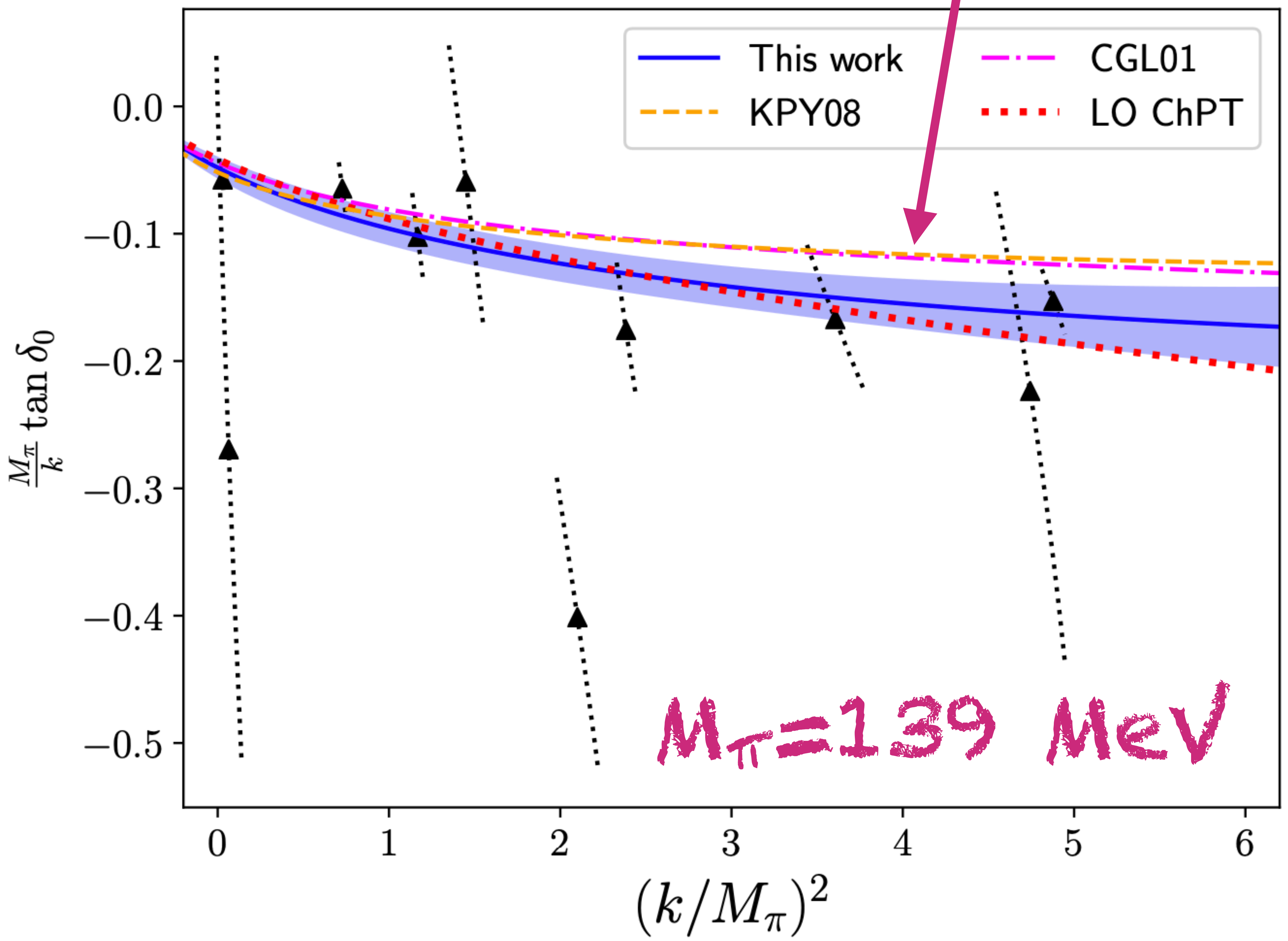

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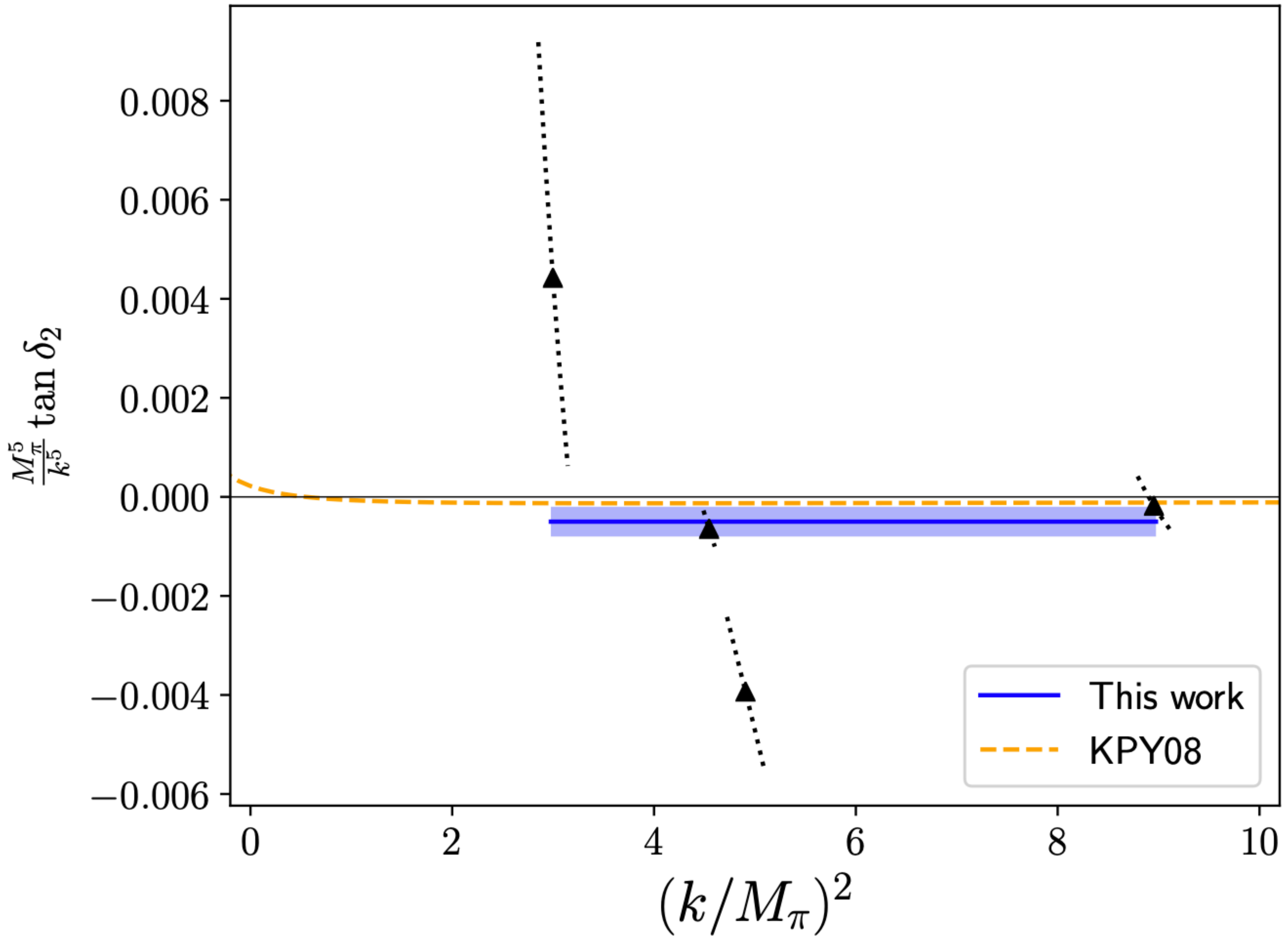

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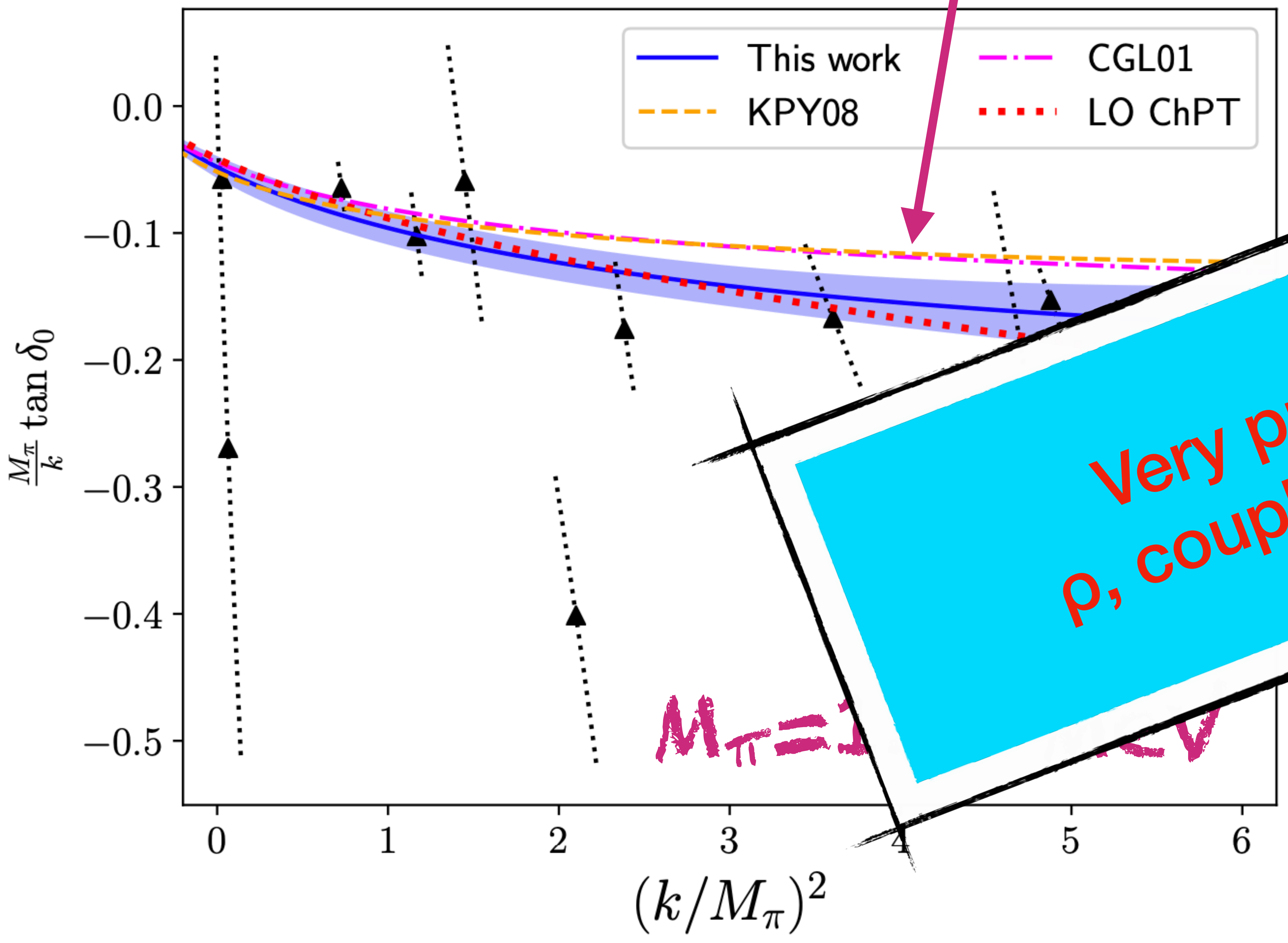

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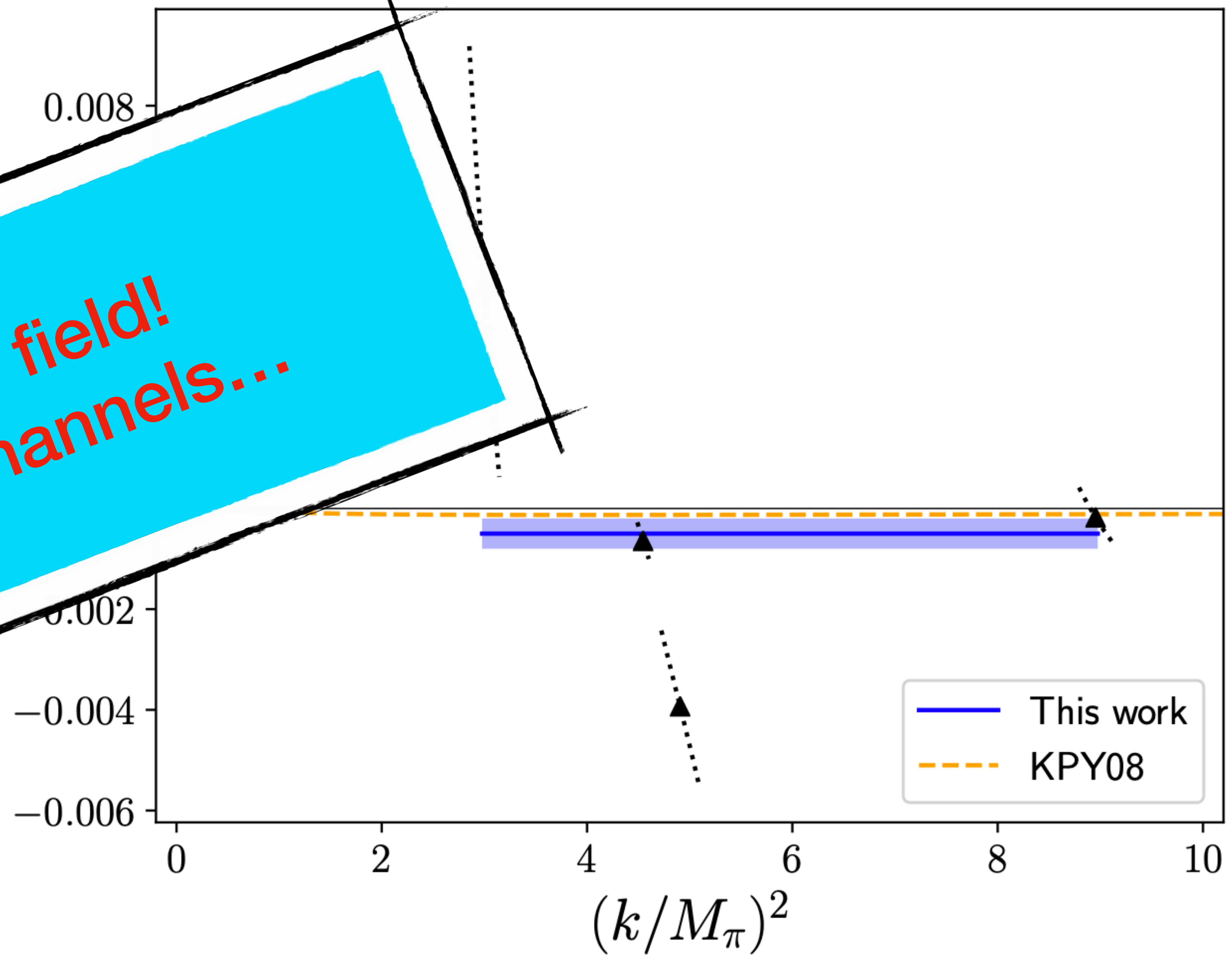
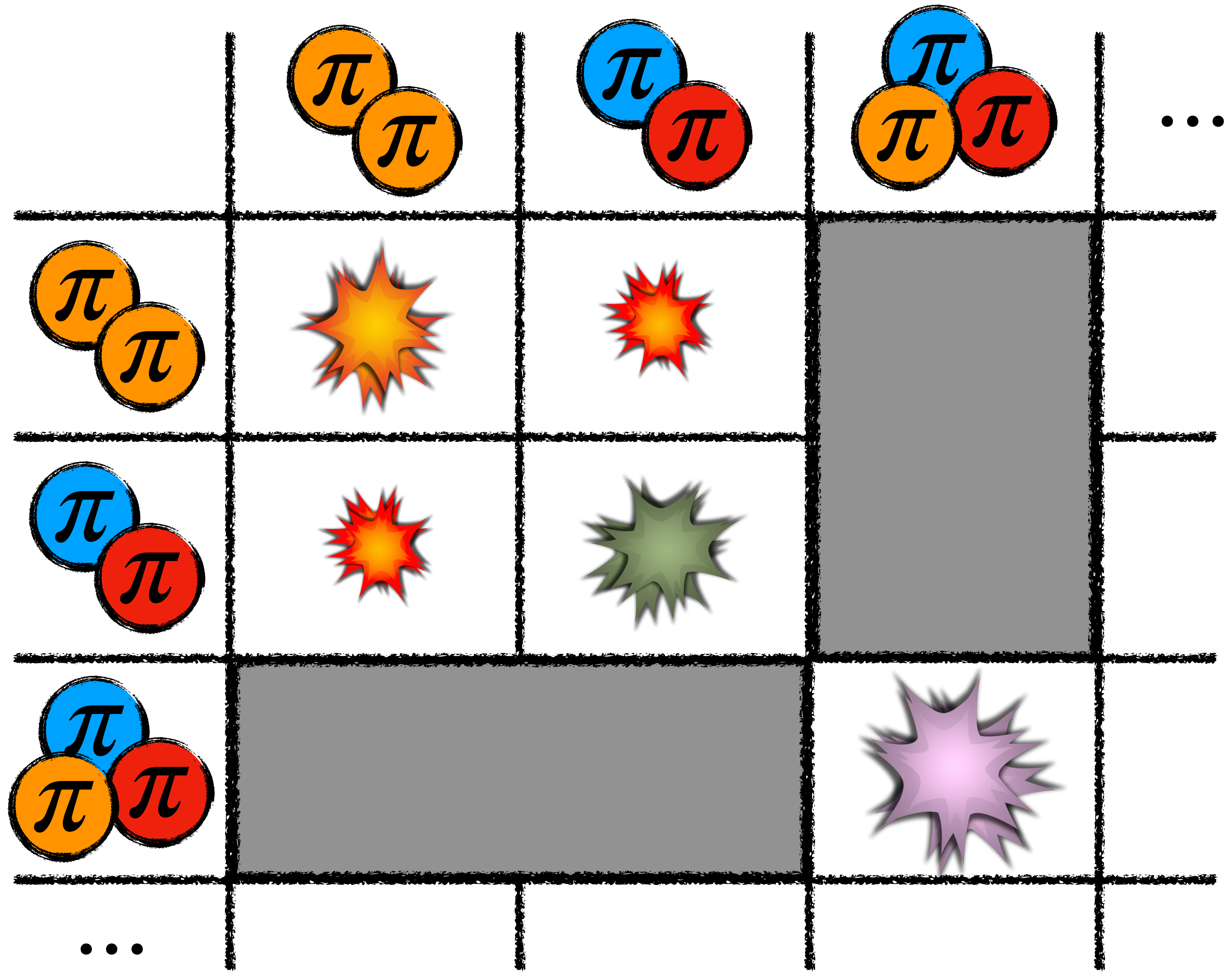


Very prolific field!
ρ, coupled channels…

[Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC)]

Three particles in finite volume



Quantization Condition (I)

Three-particle Quantization Condition for identical scalars with G-parity

$$\det\left[\mathscr{K}_{df,3}(E) + F_3^{-1}(E, \overrightarrow{P}, L)\right] = 0$$

[Hansen, Sharpe]

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Recovering the physical amplitude requires a further step

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Quantization Condition(II)

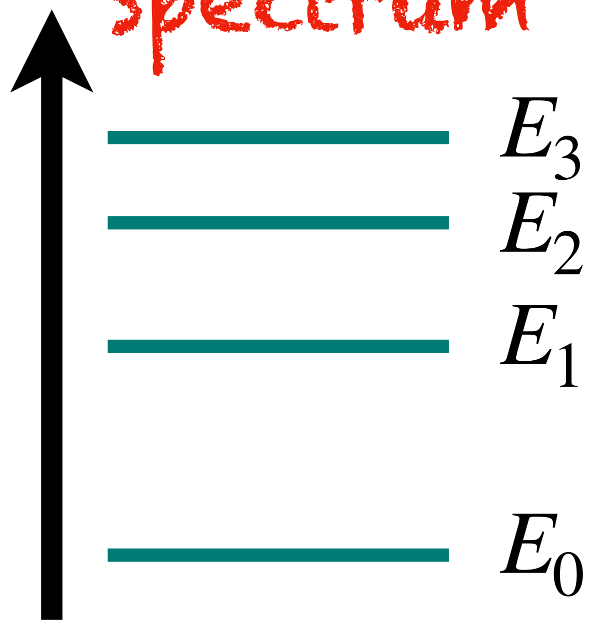
1. $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ parametrize interactions. They can be obtained from the spectrum

[Hansen, Sharpe]

Alternative approaches:
[Mai, Döring], [Hammer, et al.]

Quantization Condition(II)

2π and 3π Spectrum



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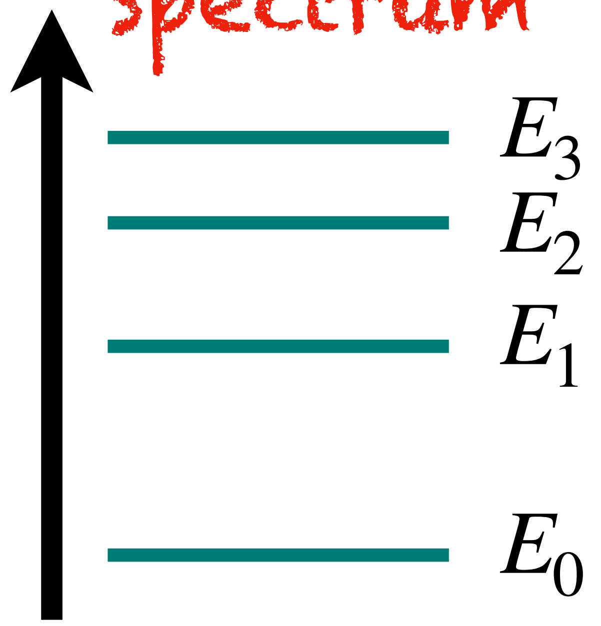

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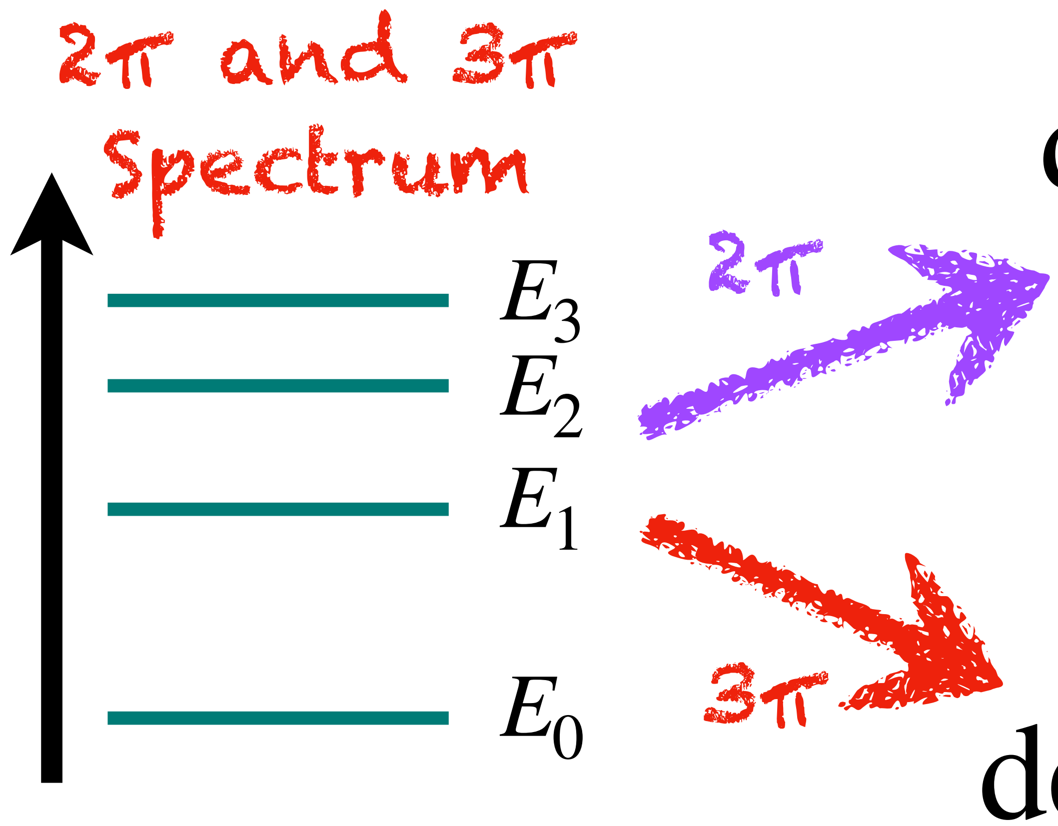

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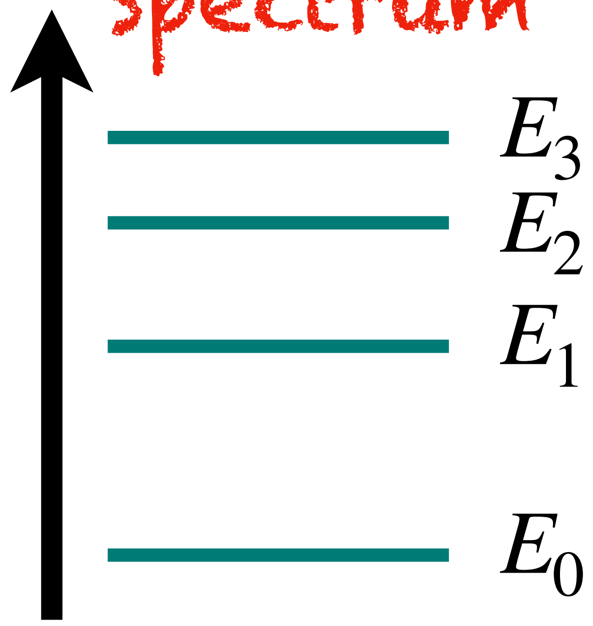

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Physical 3->3 amplitude

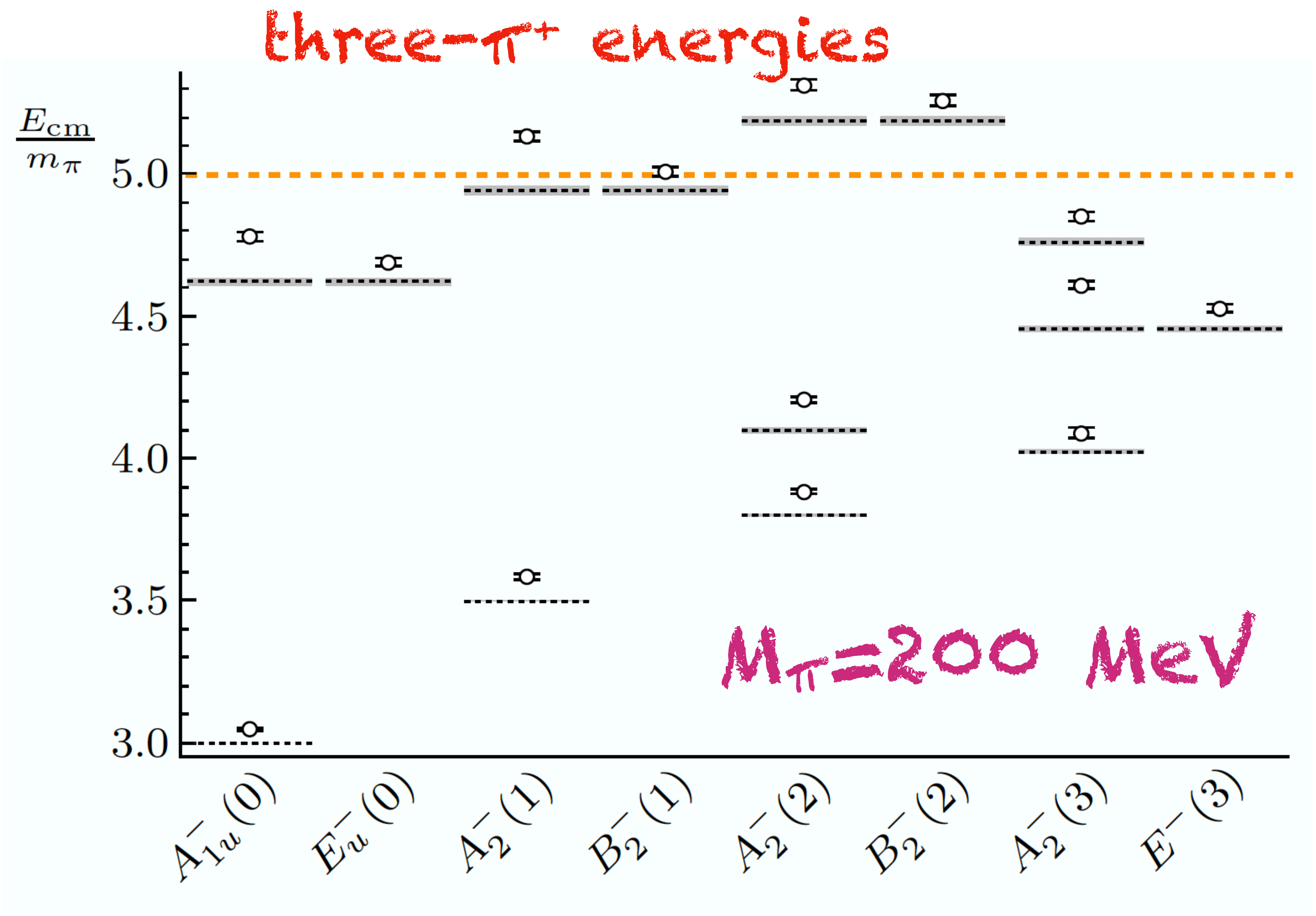
$$\mathcal{K}_2, \mathcal{K}_{df,3} \xrightarrow{\text{Integral equations}} \mathcal{M}_3$$

2. **Solve integral equations to obtain the physical three-to-three amplitude**

Derived by [Hansen, Sharpe]

Solved in [Briceño et al] , [Hansen et al.], [Jackura et al.]

two & three-pion spectrum



[Hörz,Hanlon (PRL)]

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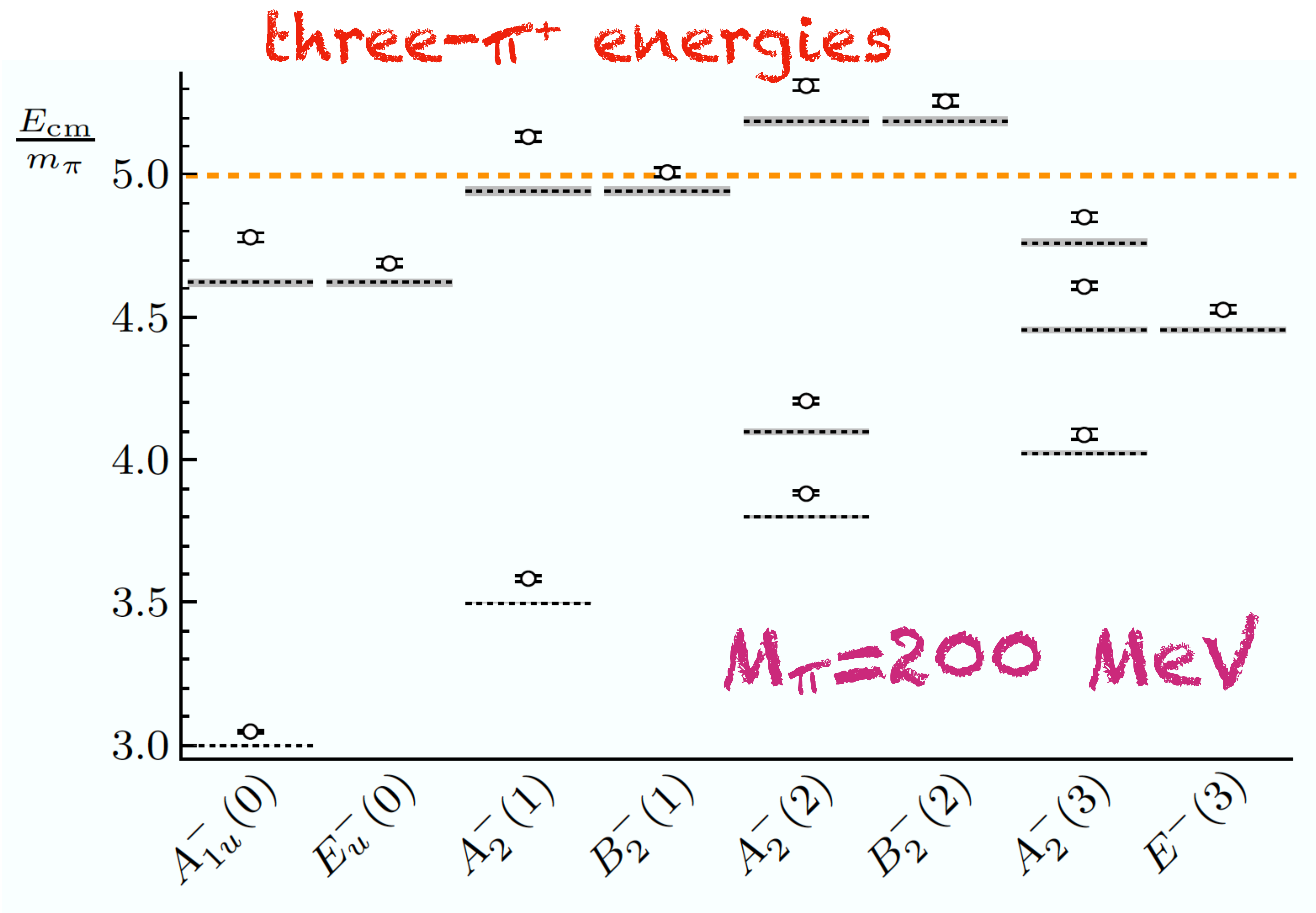

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$I = 3$ three-pion scattering amplitude from lattice QCD

Tyler D. Blanton,[1,*] Fernando Romero-López,[2,†] and Stephen R. Sharpe[1,‡]

[1] *Physics Department, University of Washington, Seattle, WA 98195-1560, USA*
[2] *Instituto de Física Corpuscular, Universitat de València and CSIC, 46980 Paterna, Spain*

(Dated: February 4, 2020)

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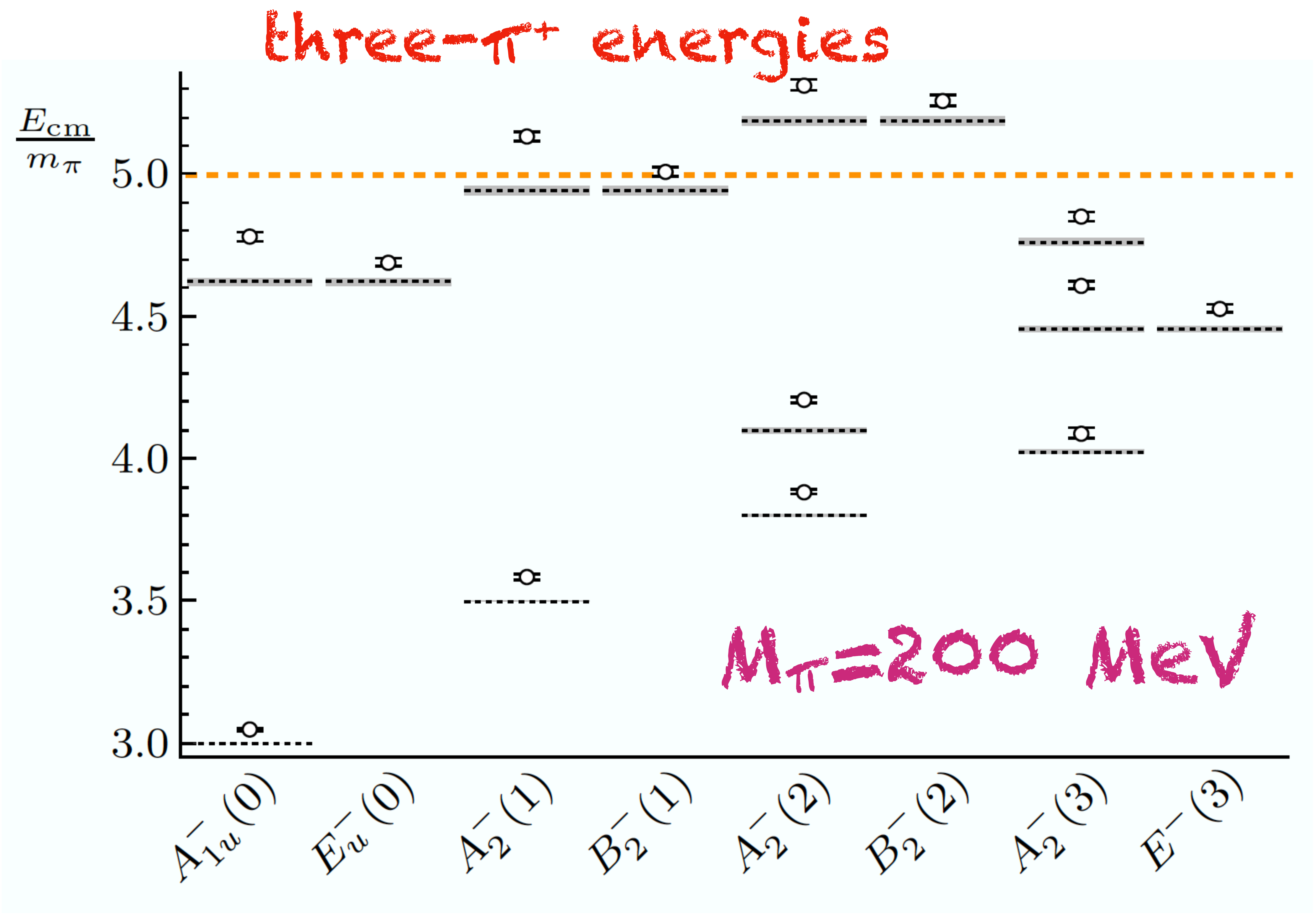

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First full analysis of the finite-volume spectrum of $2\pi^+$ and $3\pi^+$!

Fit results

- Parametrize $\mathcal{K}_{df,3}$ including only s-wave interactions:

$$\frac{q}{M}\cot\delta_0 = \frac{\sqrt{s}M}{s - z_2^2}\left(B_0 + B_1 q^2 + \cdots\right)$$

$$\mathcal{K}_{df,3} = \mathcal{K}_{df,3}^{iso,0} + \mathcal{K}_{df,3}^{iso,1}\left(\frac{s - 9M^2}{9M^2}\right)$$

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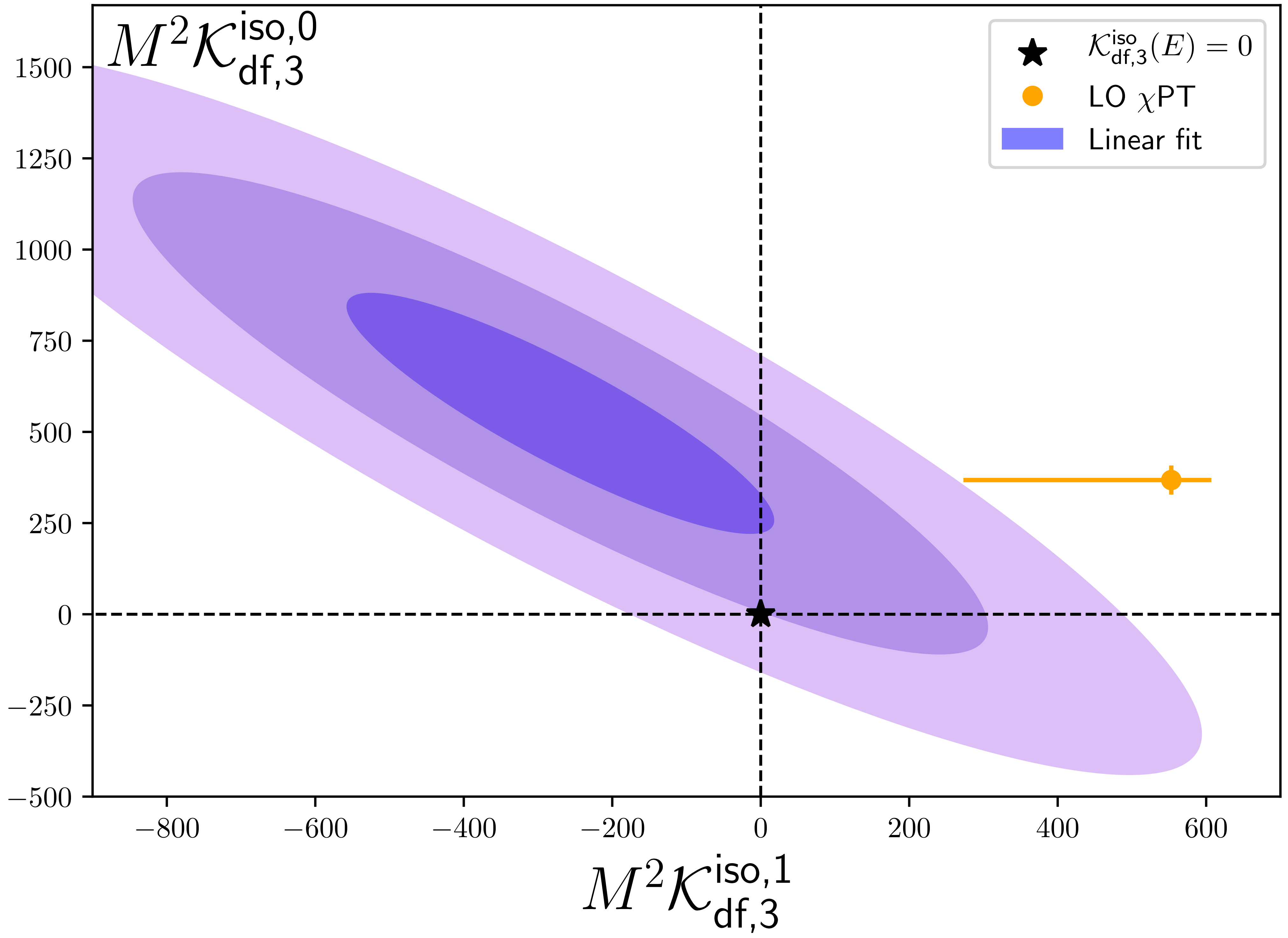
Fit	B_0	B_1	z_2^2/M^2	$M^2\mathcal{K}^{\text{iso},0}_{\text{df},3}$	$M^2\mathcal{K}^{\text{iso},1}_{\text{df},3}$	χ^2/dof	Ma_0	$M^2 r a_0$
5	-11.1(7)	-2.4(3)	1 (fixed)	550(330)	-280(290)	26.04/(22-4)	0.090(5)	2.57(8)



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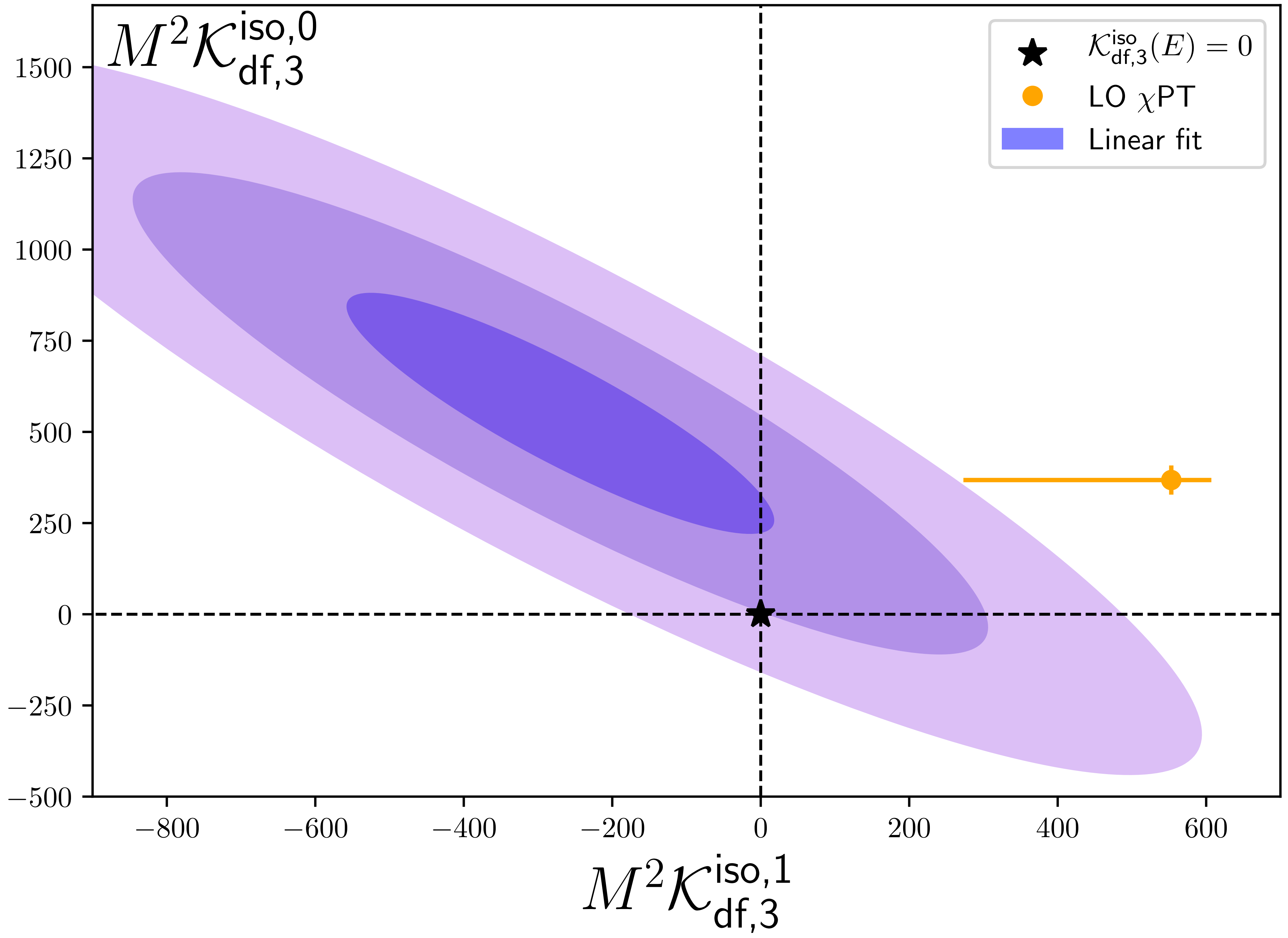

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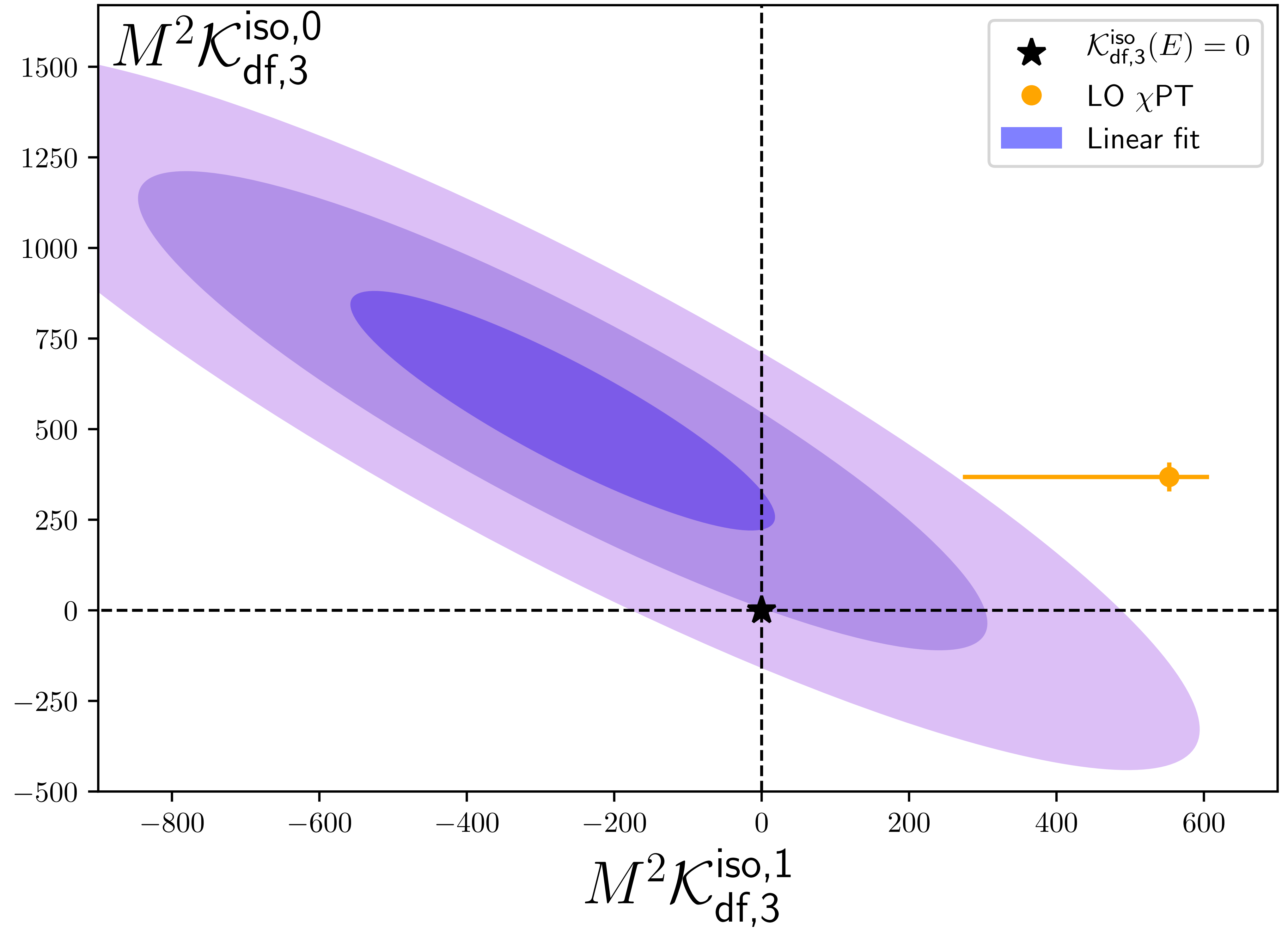

1. 2σ evidence for $\mathcal{K}_{df,3} \neq 0$.

2. Some tension with ChPT.

Chiral dependence

- On a later article, the chiral dependence of $\mathcal{K}_{df,3}$ has been studied, including physical pions.
  [Fischer, Kostrzewa, Liu, <u>FRL</u>, Ueding, Urbach (ETMC)]

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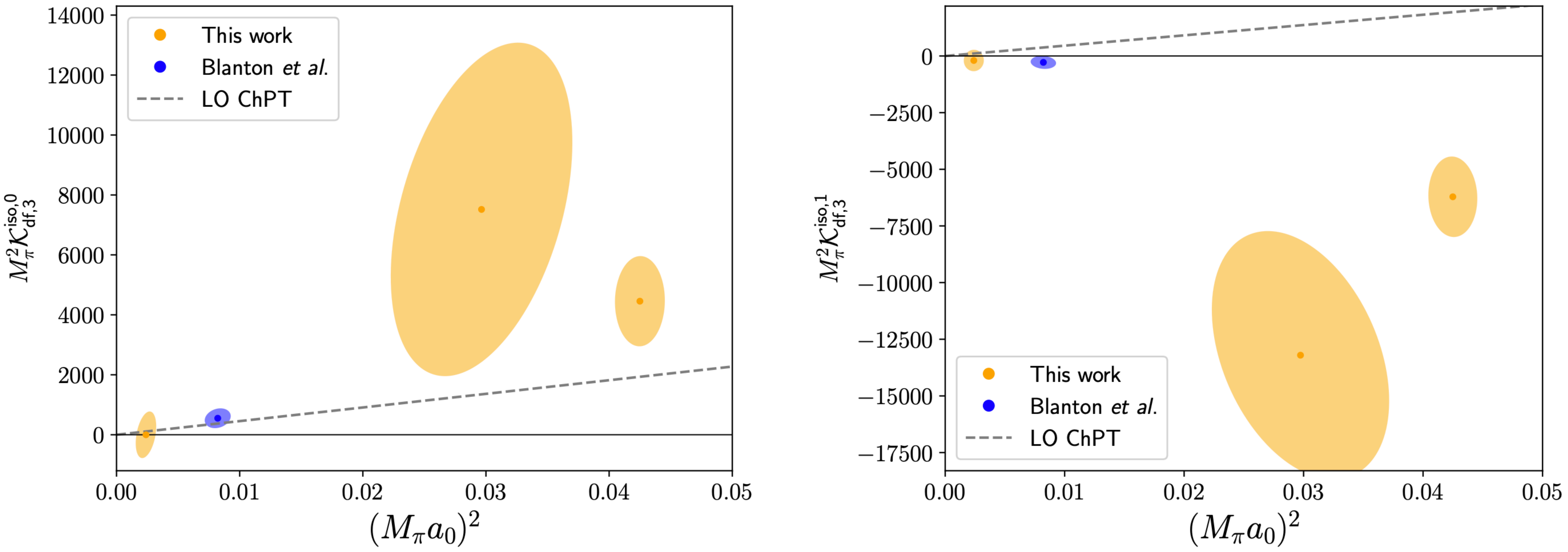

(a) $\mathcal{K}^{\text{iso},0}_{\text{df},3}$ (b) $\mathcal{K}^{\text{iso},1}_{\text{df},3}$

see also other studies:
[Mai et al., Culver et al.]

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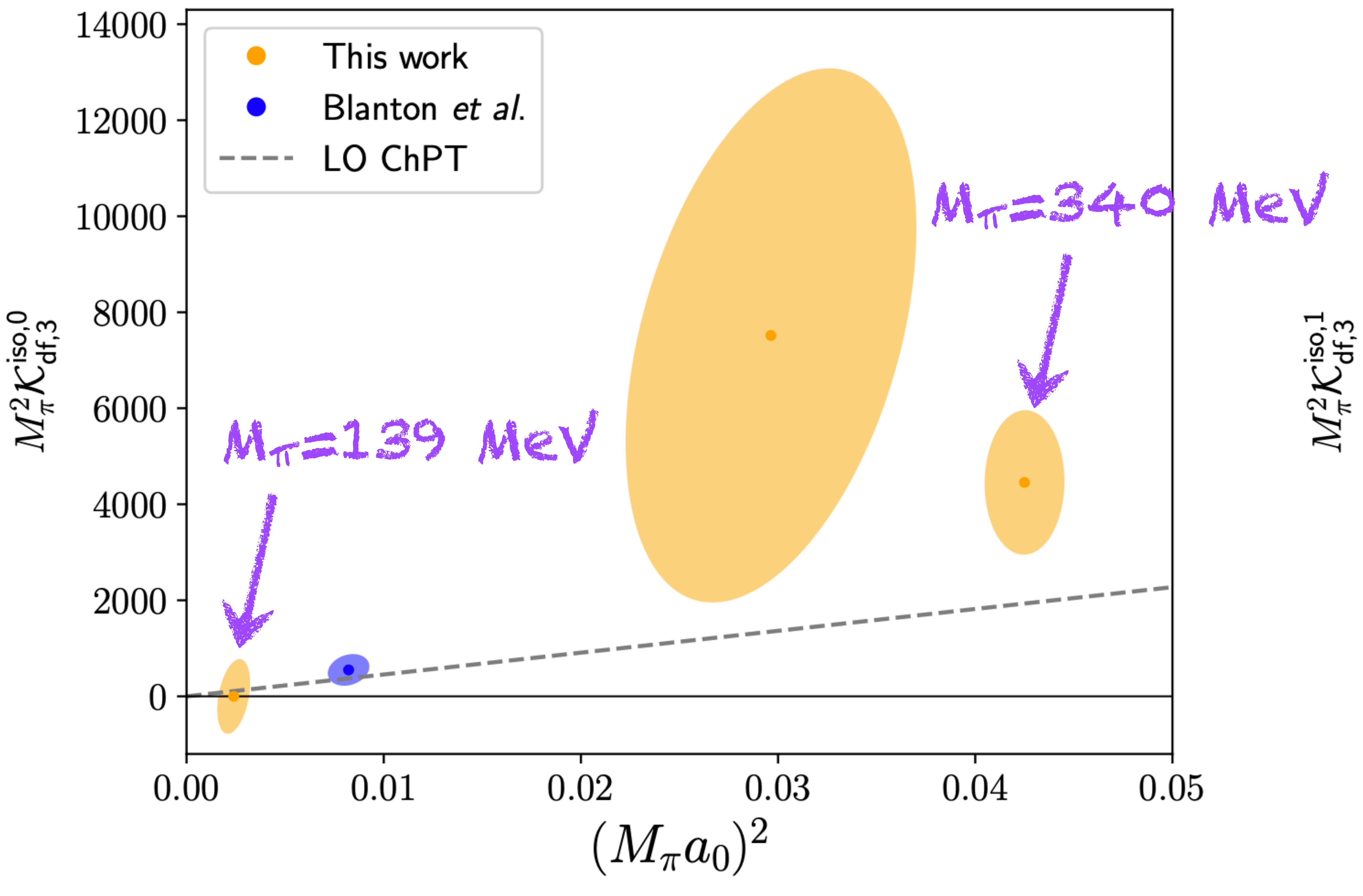

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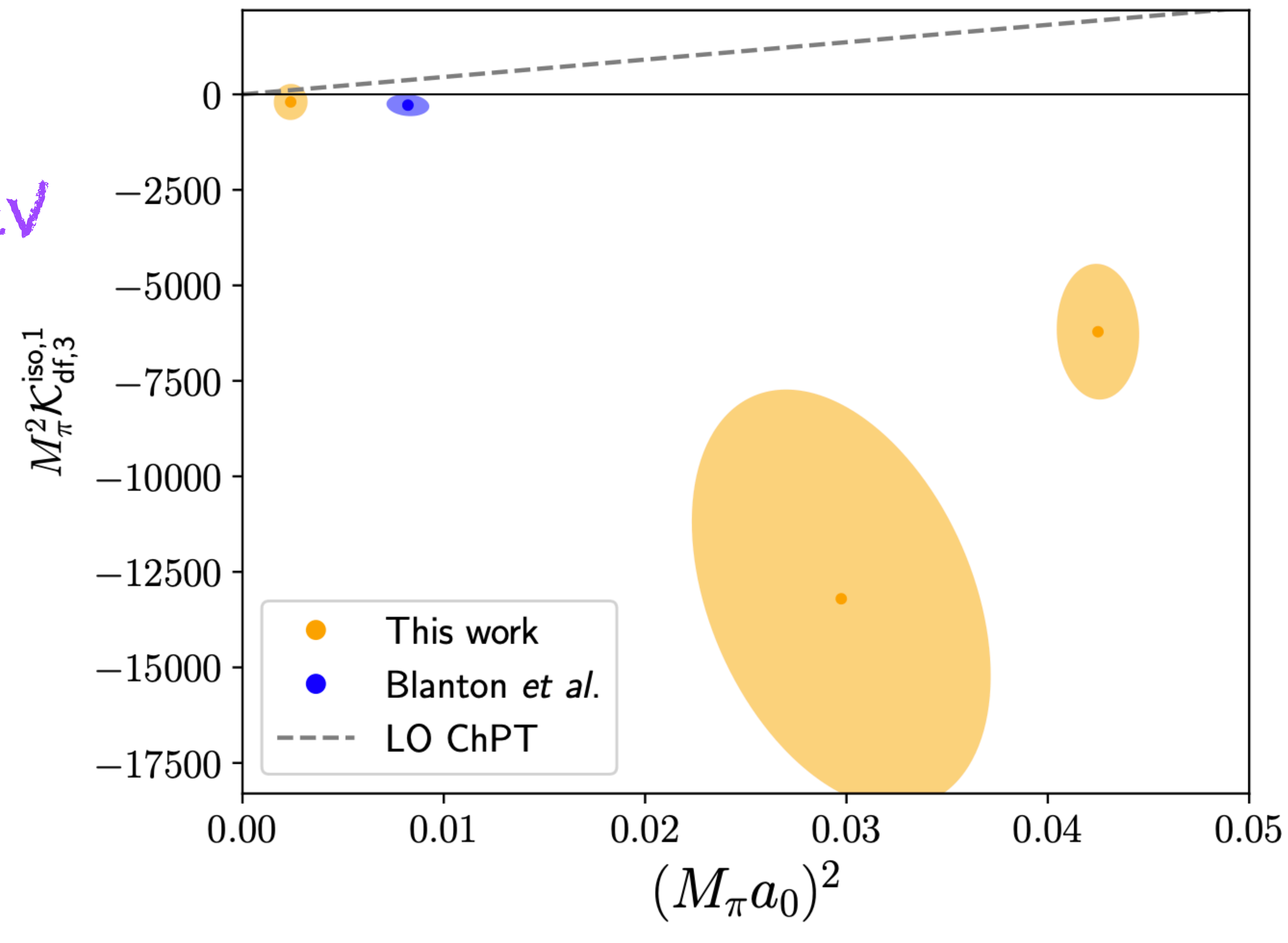

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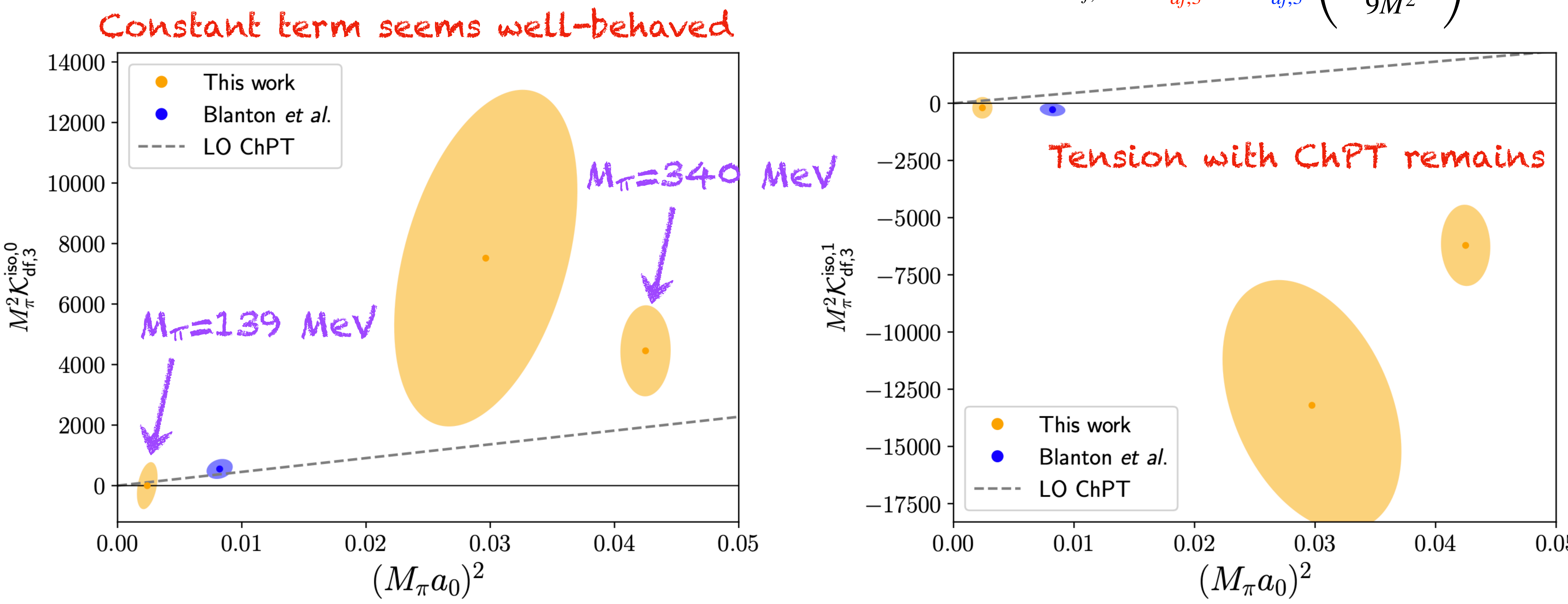

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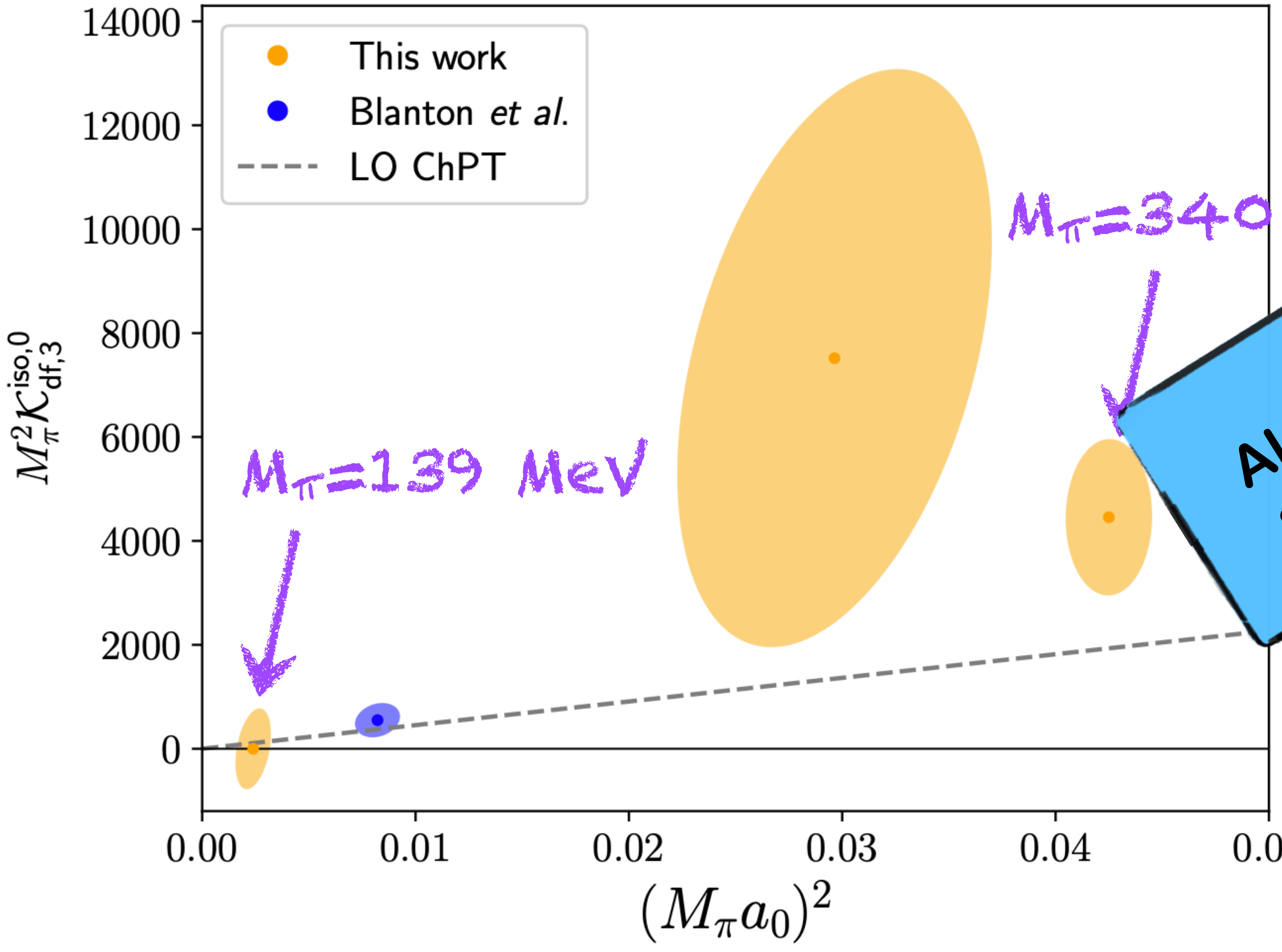
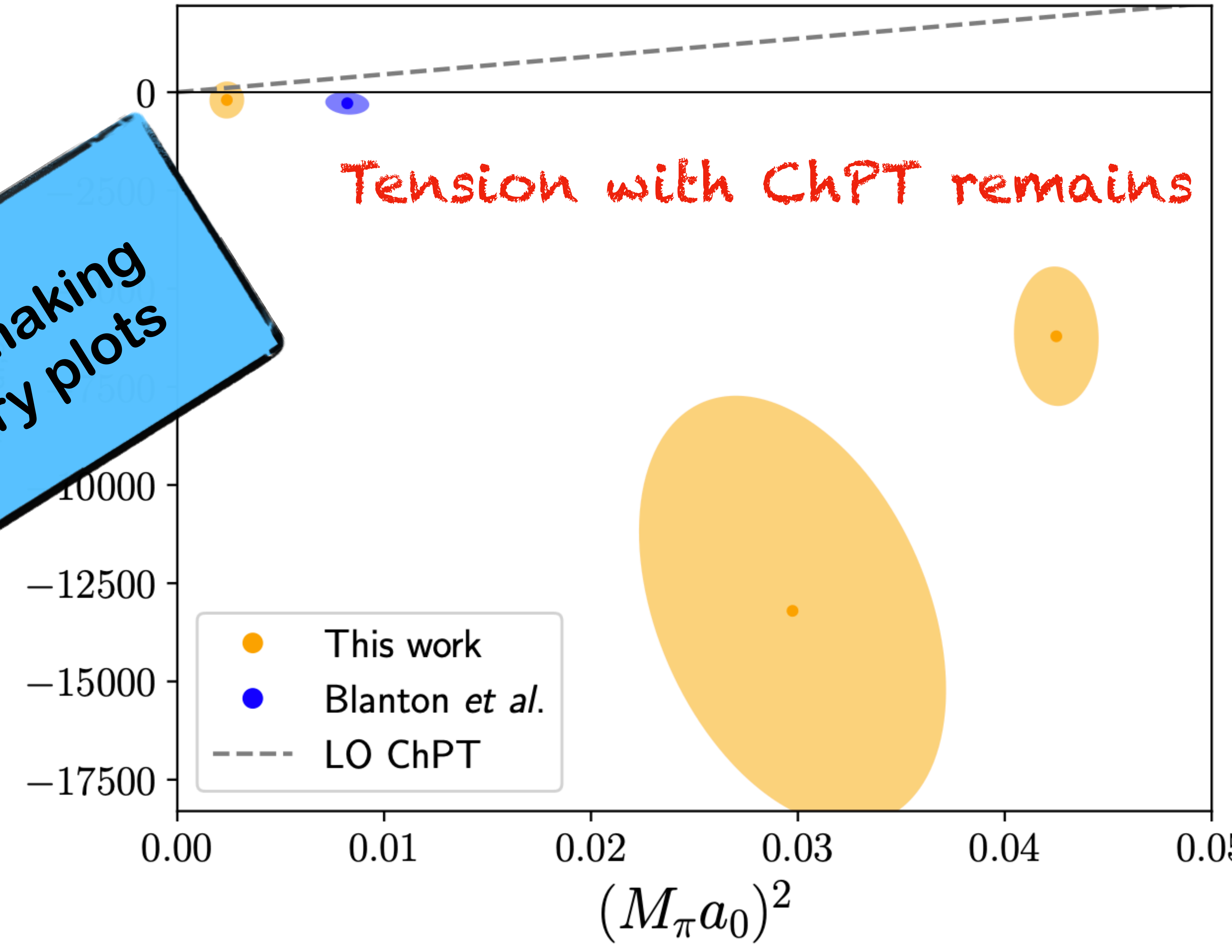


(a) $\mathcal{K}^{\text{iso},0}_{\text{df},3}$

Already making summary plots

Tension with ChPT remains



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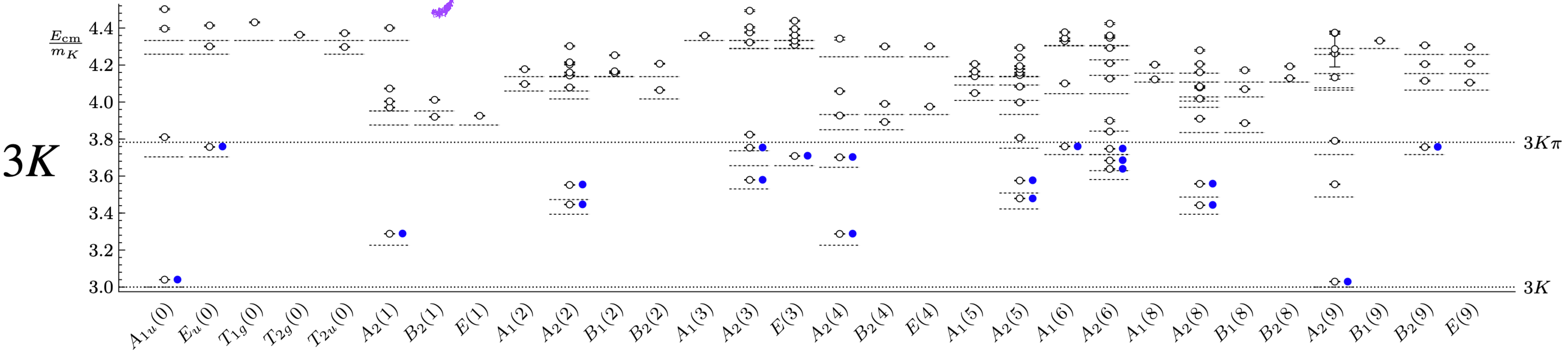
Two- and three-kaons

- Other simple systems can also be studied: $2K^+$ & $3K^+$
- Many energy levels that allow for s- and d-wave interactions to be extracted!

Preliminary!



[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe (in preparation)]

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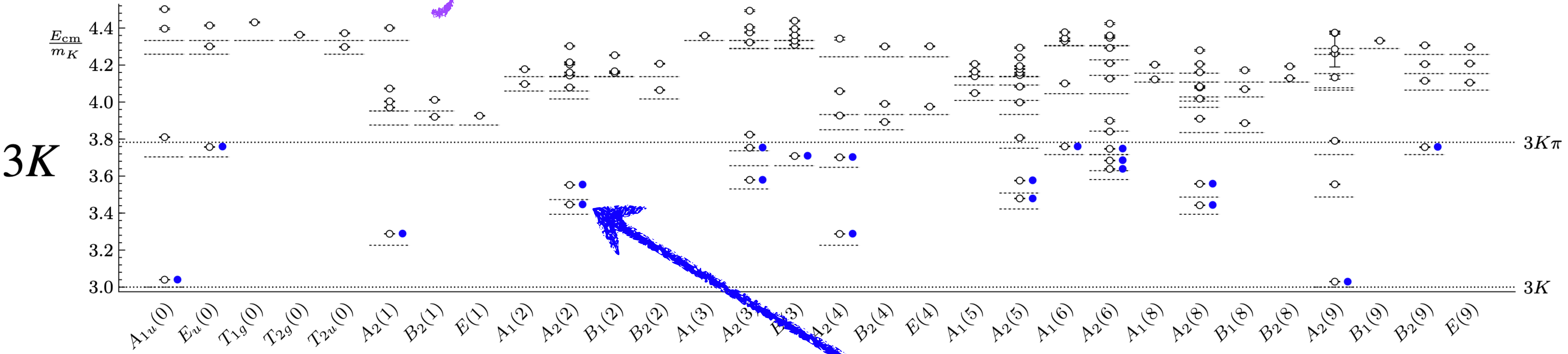

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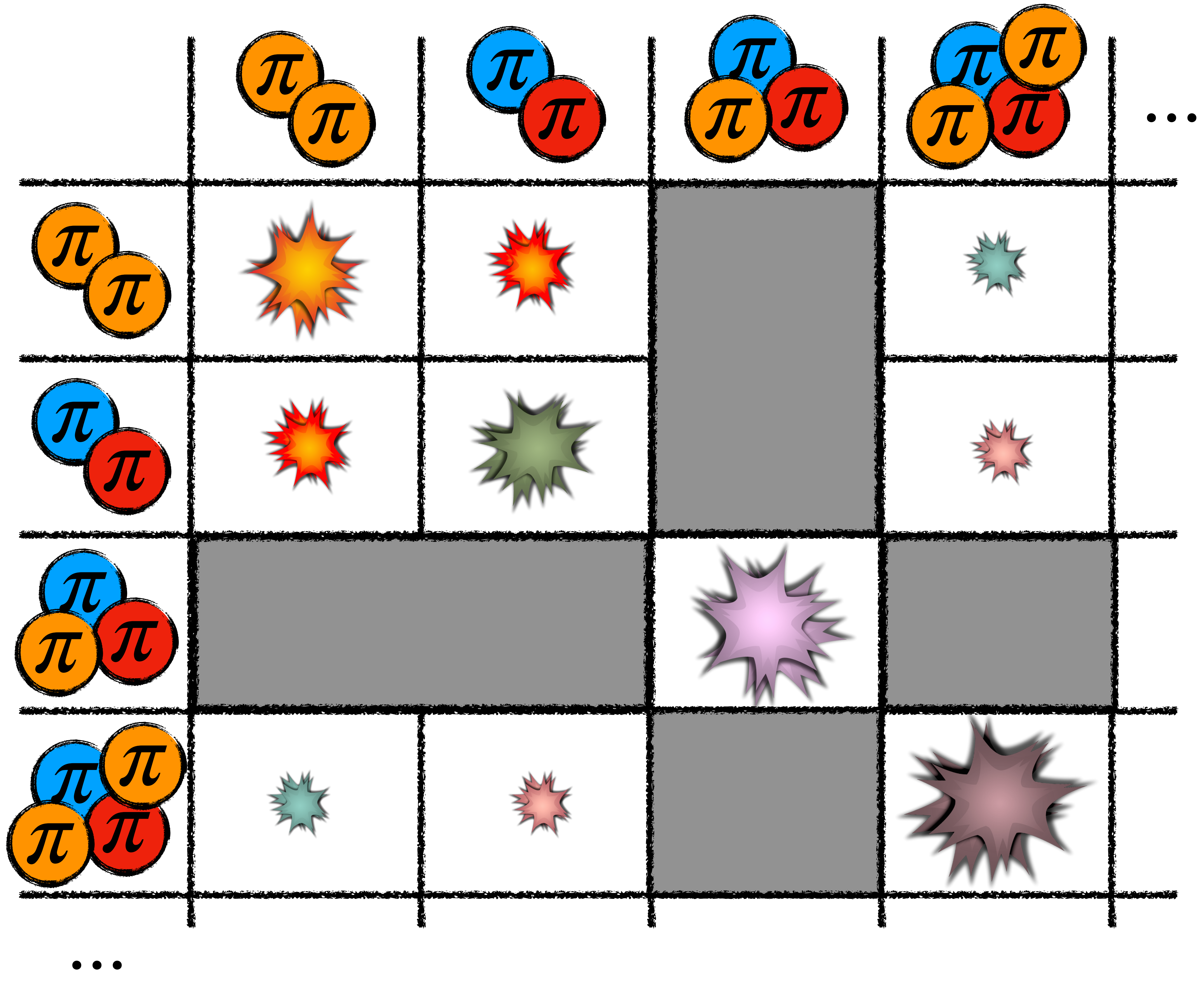
Outlook



Next steps

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2. Generalizing the formalism for generic two- and three- particle systems, (e.g. nucleons, Roper resonance)

3. Formalism for three-particle decays, such as $K \to 3\pi\,, \gamma \to 3\pi$

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Thanks!