Perturbative gauge invariants in the Hamiltonian formalism for spherically symmetric backgrounds

Andrés Mínguez-Sánchez[1].

[1]Instituto de Estructura de la Materia (**IEM**).

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CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

Introduction and motivation

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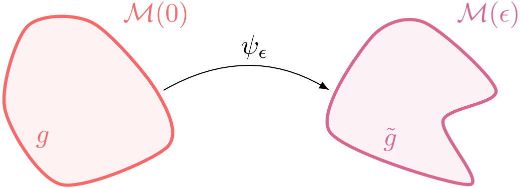
- A Hamiltonian formalism constitute the basis for quantum studies.

Objetive

- Apply a perturbative Hamiltonian formalism to spherically symmetric models, then specialize it for the Schwarzschild case.
- For a quantum treatment of the perturbations, pioneering works in loop quantum cosmology suggest starting with the black hole interior.

General framework for perturbations



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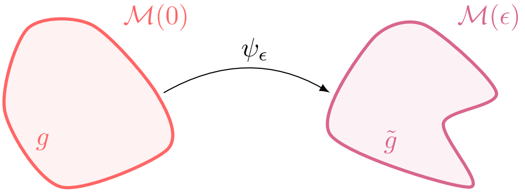

Perturbations are defined by the map ψ_ϵ (its choice is **not unique**). Hierarchy is determined by ϵ. A perturbative quantity, $\tilde{g}(\epsilon)$, is defined on $\mathcal{M}(0)$ as

$$\psi_\epsilon^* \tilde{g}(\epsilon) = g + \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \Delta_\psi^n[g], \qquad \Delta_\psi^n[g] = \left. \frac{\mathrm{d}^n \psi_\epsilon^* \tilde{g}(\epsilon)}{\mathrm{d}\epsilon^n} \right|_{\epsilon=0}. \tag{1}$$

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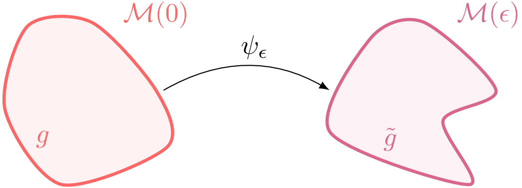

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Remark

Perturbative gauge-invariants do not depend on the map election.

First-order perturbations: Hamiltonian variables

The perturbed canonical variables, up to first order, for any spherically symmetric background, can be expressed as

$$\mathbf{g} = g + \epsilon h + O(\epsilon^2), \qquad \mathbf{P} = \Pi + \epsilon p + O(\epsilon^2). \tag{2}$$

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Depending on their parity, the harmonics distinguish between

$$\textbf{Polar:} \quad \mathcal{P} \xrightarrow{\mathbf{P}} (-1)^l \mathcal{P}, \qquad \textbf{Axial:} \quad \mathcal{A} \xrightarrow{\mathbf{P}} (-1)^{l+1} \mathcal{A}, \tag{3}$$

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At **first order**, axial and polar perturbations can be treated **independently**.

First-order perturbations: Hamiltonian dynamics

The dynamics of a perturbed system, up to first order, is determined by

$$S = S_0 + \frac{\epsilon^2}{2}\Delta_1^2[S] + O(\epsilon^3), \tag{4}$$

where S_0 is the background action and $\Delta_1^2[S]$ is given by

$$\frac{1}{2}\Delta_1^2[S] = \frac{1}{\kappa}\int_{\mathbb{R}} \mathrm{d}t \int_{\sigma} \mathrm{d}^3x \left(h_{ab,t}p^{ab} - C\Delta[\mathcal{H}] - B^a\Delta[\mathcal{H}_a] - \frac{N}{2}\Delta_1^2[\mathcal{H}] - \frac{N^a}{2}\Delta_1^2[\mathcal{H}_a] \right). \tag{5}$$

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Upon integrating over the two-sphere, for each perturbative mode we have:

	Variables	**Constraints**	**Hamiltonian**
Axial	$\{h_i^{l,m}, p_i^{l,m}\}_{i=1}^{2}$	$C_0^{l,m}$	$H_{\mathrm{ax}}^{l,m}$
Polar	$\{h_i^{l,m}, p_i^{l,m}\}_{i=3}^{6}$	$C_1^{l,m}, C_2^{l,m}, C_3^{l,m}$	$H_{\mathrm{po}}^{l,m}$

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For each perturbative mode consider a (mode- and background-dependent) canonical transformation,

$$\left\{h_i^{l,m}, p_i^{l,m}\right\}_{i=1}^{6} \longrightarrow \left\{Q_i^{l,m}, P_i^{l,m}\right\}_{i=1}^{6}, \tag{6}$$

such that the new perturbative variables satisfy

$$P_2^{l,m} = C_0^{l,m}, \qquad P_4^{l,m} = C_1^{l,m}, \qquad P_5^{l,m} = C_2^{l,m}, \qquad P_6^{l,m} = C_3^{l,m}. \tag{7}$$

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- ✦ After redefining the Lagrange multipliers (background and perturbations), the **Hamiltonian depends only on the two invariant pairs**.

Black hole interior: background framework

The spatial metric g and its conjugate momentum Π are defined as

$$
\begin{aligned}
g &= \frac{p_b^2(t)}{L_o^2|p_c(t)|}\mathrm{d}x^2 + |p_c(t)|(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \\
\Pi &= -2\frac{L_o^2}{p_b^2(t)}\Omega_b(t)|p_c(t)|\sin\theta\partial_x^2 - \frac{\Omega_b(t)+\Omega_c(t)}{|p_c(t)|}\left(\sin\theta\partial_\theta^2 + \csc\theta\partial_\phi^2\right),
\end{aligned}
\tag{8}
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where $\Omega_b = bp_b/(\gamma L_o)$, $\Omega_c = cp_c/(\gamma L_o)$, and L_o is a fiducial length under the assumption of a **compact** spatial topology, $\sigma_o = S_o^1 \times S^2$.

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Following spatial integration and appropriate gauge fixing, the symplectic structure and dynamics are determined by

$$\{b, p_b\}_\mathrm{B} = \gamma, \quad \{c, p_c\}_\mathrm{B} = 2\gamma, \quad \tilde{H}_\mathrm{B} = -L_o\left[\Omega_b^2 + \frac{p_b^2}{L_o^2} + 2\Omega_b\Omega_c\right]. \tag{9}$$

Black hole interior: first-order perturbations

Using $\mathfrak{n} = (n, l, m)$ to simplify the notation of the mode labels, first-order perturbations can be expanded as

$$h = \sum_{\mathfrak{n},\lambda} h_6^{\mathfrak{n},\lambda} Y_l^m Q_{n,\lambda} \mathrm{d}x^2 + \sum_{\mathfrak{n},\lambda} 2\left[h_5^{\mathfrak{n},\lambda} Z_l{}^m{}_A - h_1^{\mathfrak{n},\lambda} X_l{}^m{}_A\right] Q_{n,\lambda} \mathrm{d}x \mathrm{d}x^A$$
$$+ \sum_{\mathfrak{n},\lambda} \left[h_4^{\mathfrak{n},\lambda} Z_l{}^m{}_{AB} + h_3^{\mathfrak{n},\lambda} Y_l{}^m{}_{AB} + h_2^{\mathfrak{n},\lambda} X_l{}^m{}_{AB}\right] Q_{n,\lambda} \mathrm{d}x^A \mathrm{d}x^B,$$

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Black hole interior: gauge invariants

After canonical transformations and redefinition of the Lagrange multipliers, the dynamics of each perturbative mode is given by

$$\begin{aligned}\mathbf{H}^{\mathfrak{n},\lambda} = {} & b_{\rm a}(\hat{l}_{\rm a})\left([P_1^{\mathfrak{n},\lambda}]^2 + [k_{\rm a}^2 + s_{\rm a}(\hat{l}_{\rm a})][Q_1^{\mathfrak{n},\lambda}]^2\right) \\ & + b_{\rm p}(\hat{l}_{\rm p})\left([P_3^{\mathfrak{n},\lambda}]^2 + [k_{\rm p}^2 + s_{\rm p}(\hat{l}_{\rm p})][Q_3^{\mathfrak{n},\lambda}]^2\right).\end{aligned} \tag{11}$$

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where b_{a}, b_{p}, s_{a}, and s_{p} are **background-dependent** coefficients that depend **only** on the modes labels through

$$k_{\mathrm{a}}^2 = \frac{4\pi^2}{L_o^2}n^2 + (l+2)(l-1), \quad k_{\mathrm{p}}^2 = \frac{l(l+1)}{(l+2)(l-1)}\left(\frac{l^2+l-6}{l^2+l+2}\frac{4\pi^2}{L_o^2}n^2 + l(l+1)\right),$$

$$\hat{l}_{\mathrm{a}} = \frac{1}{k_{\mathrm{a}}}\sqrt{(l+2)(l-1)}, \qquad \hat{l}_{\mathrm{p}} = \frac{1}{k_{\mathrm{p}}}\frac{l(l+1)}{\sqrt{(l+2)(l-1)}}.$$

Conclusions

- With our method, we can **identify** and **handle** the perturbative gauge invariants for any spacetime with a spherically symmetric background.
- The Hamiltonian formulation of the black hole interior leads to a **loop** quantization of the background, combined with a (essentially **unique**) Fock representation for the perturbations.

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- With our method, we can **identify** and **handle** the perturbative gauge invariants for any spacetime with a spherically symmetric background.
- The Hamiltonian formulation of the black hole interior leads to a **loop** quantization of the background, combined with a (essentially **unique**) Fock representation for the perturbations.

Future results

- We are working on extending these results to the **exterior geometry**.
- We are aiming to relate our invariants to more conventional ones for black holes. Starting with the axial modes is the most reasonable approach.
- The **final stage** of the coalescence of supermassive black holes (ringdown phase) is a great scenario to apply this study.