Curvature Wave Equations and Ringdown Non-linearities

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Based on: [*Phys.Rev.D* 109 (2024) 4, 044048], and [arXiv:2408.13557].
With: Vitor Cardoso, Jaime Redondo-Yuste and Gowtham Rishi Mukkamala.

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Introduction

—— · *Nonlinear dynamical structure of GR* · ——

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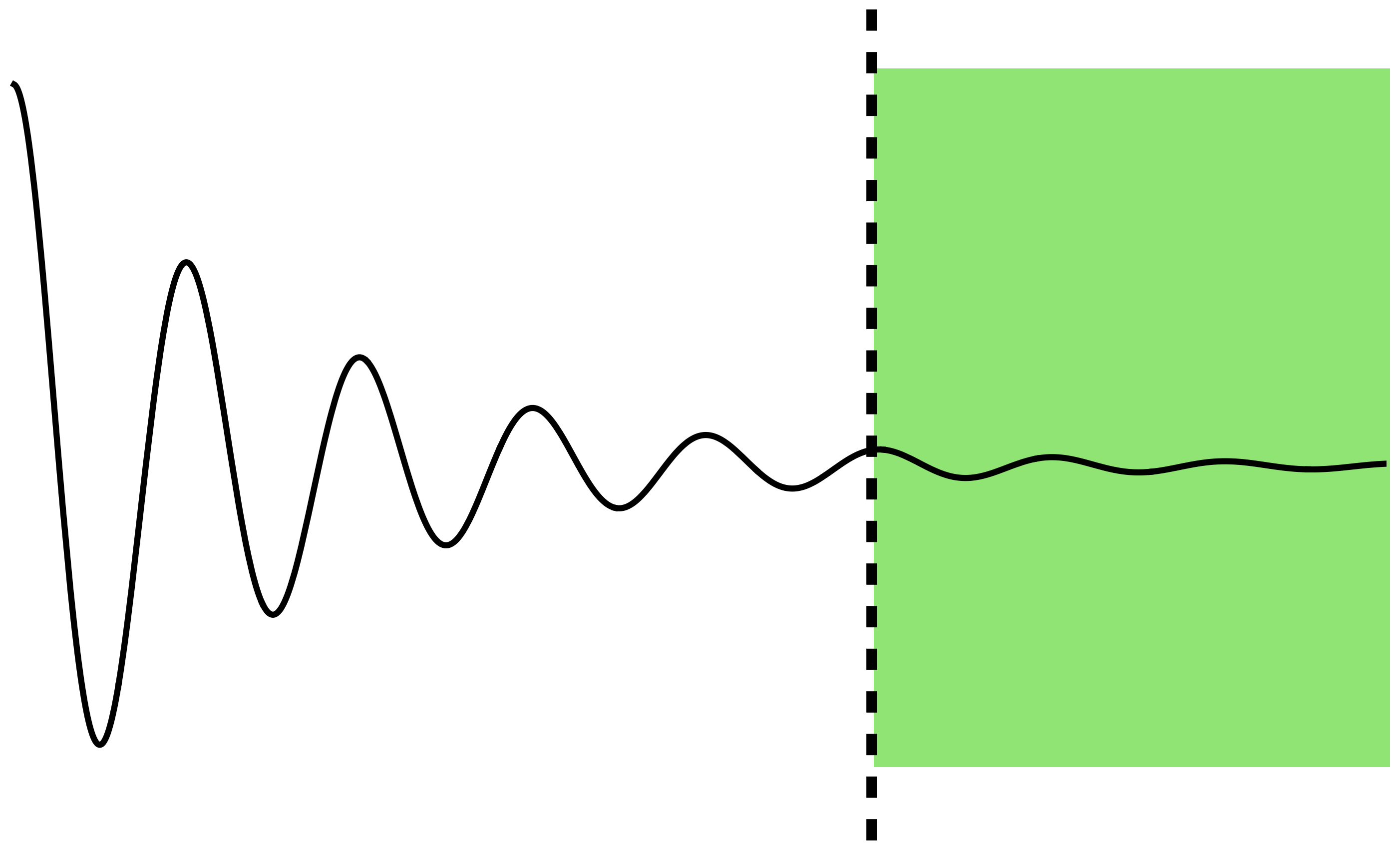
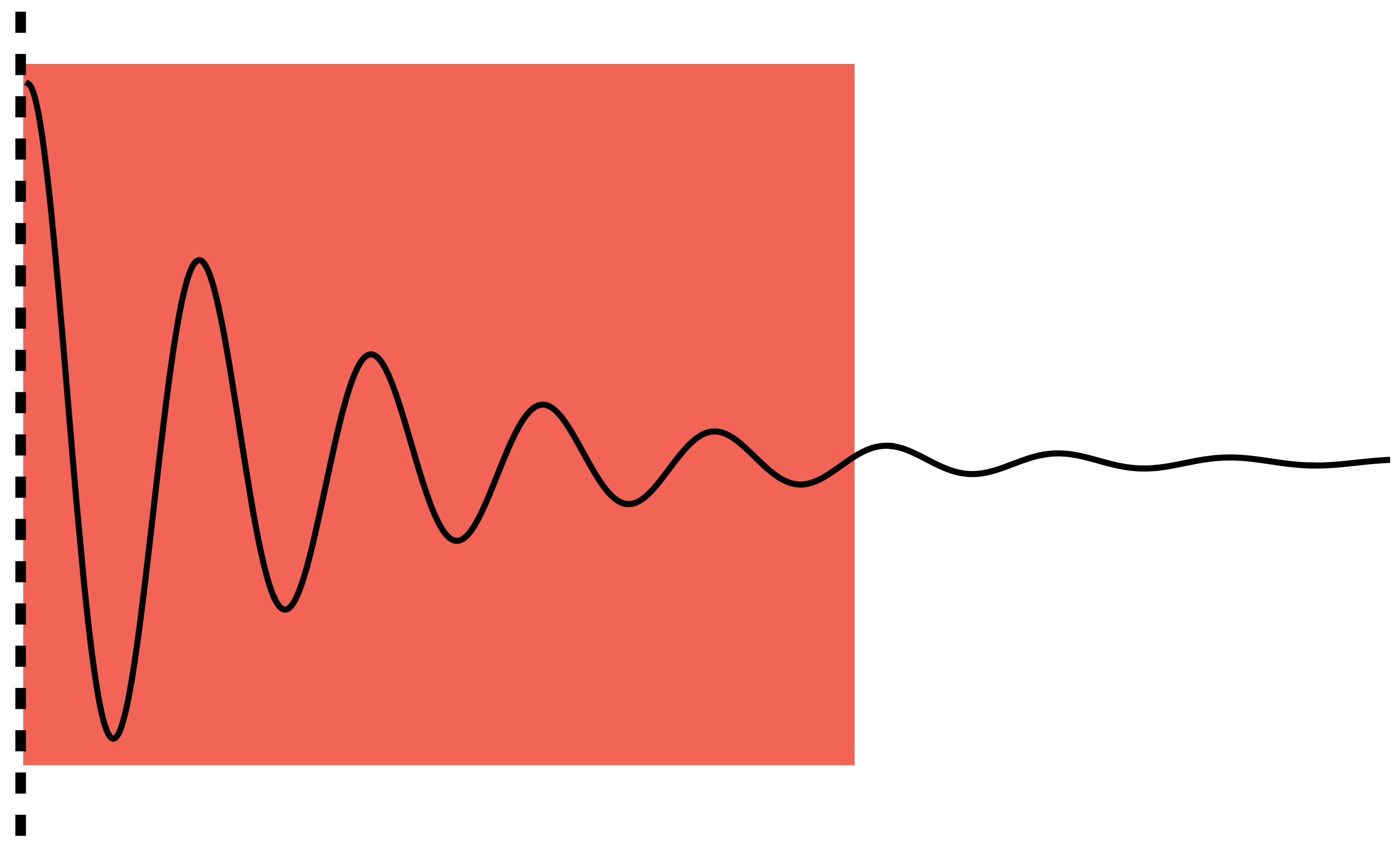
- Starting time problem



$t - t_{merger} \geq 15\,M$



$t - t_{merger} \sim 0$

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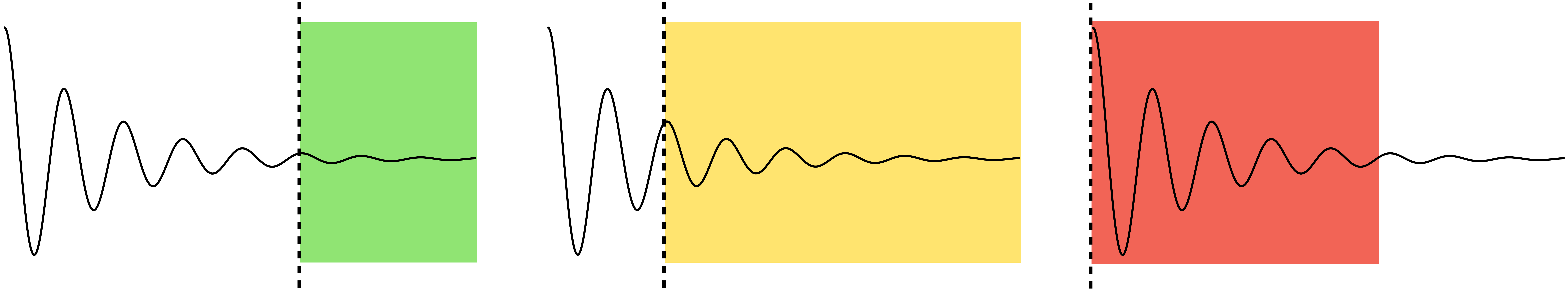

$t - t_{merger} \geq 15\,M$ $\quad$ $t - t_{merger} \in (0, 15\,M)$ $\quad$ $t - t_{merger} \sim 0$

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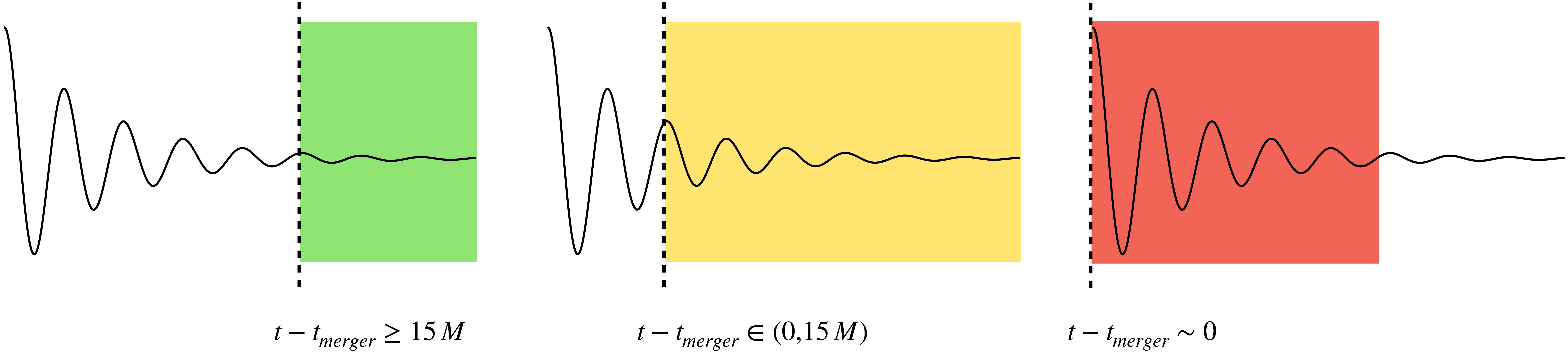
- Goal: Nonlinear RD [LISA Waveform WG 2311.01300 '23]



State of the Art (part of)

- Quadratic fluctuations

[Gleiser+ ’96, Campanelli+ ’99]

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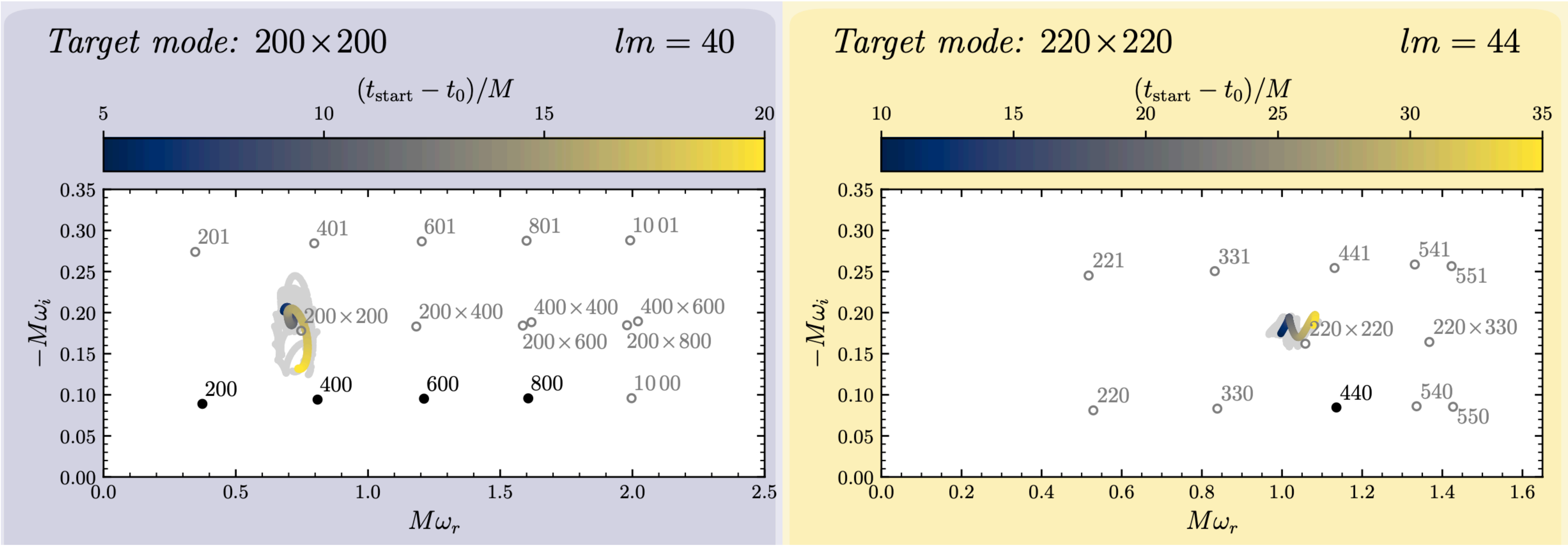

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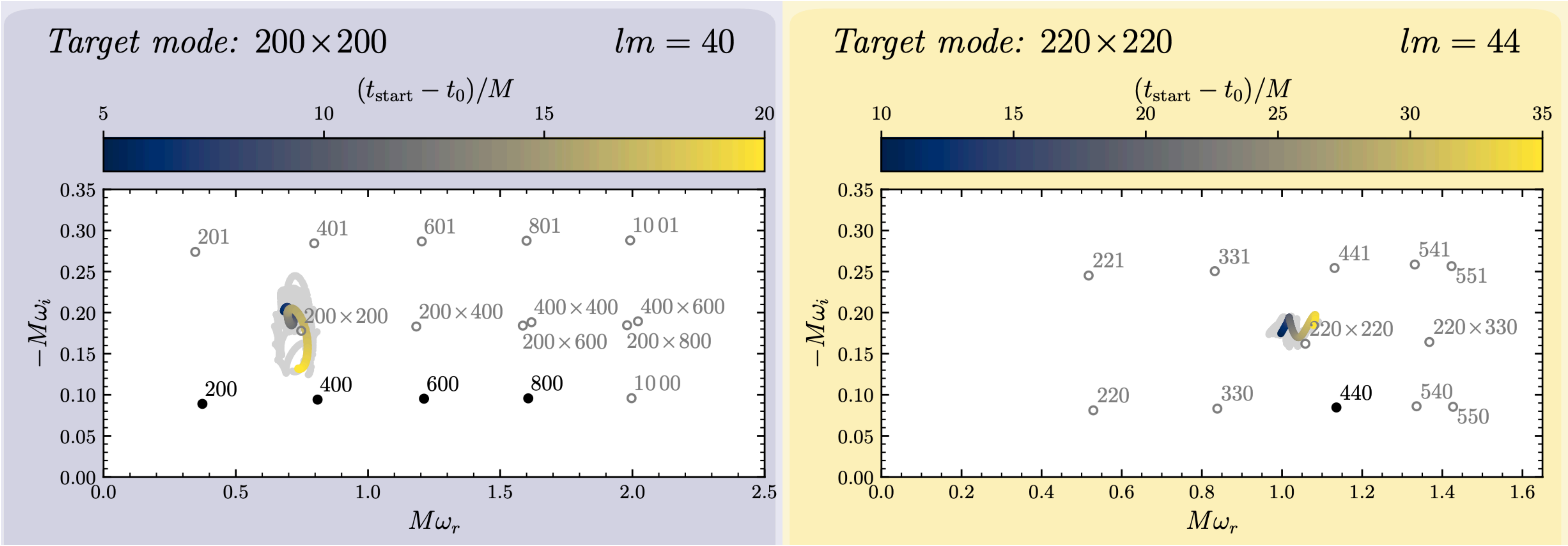
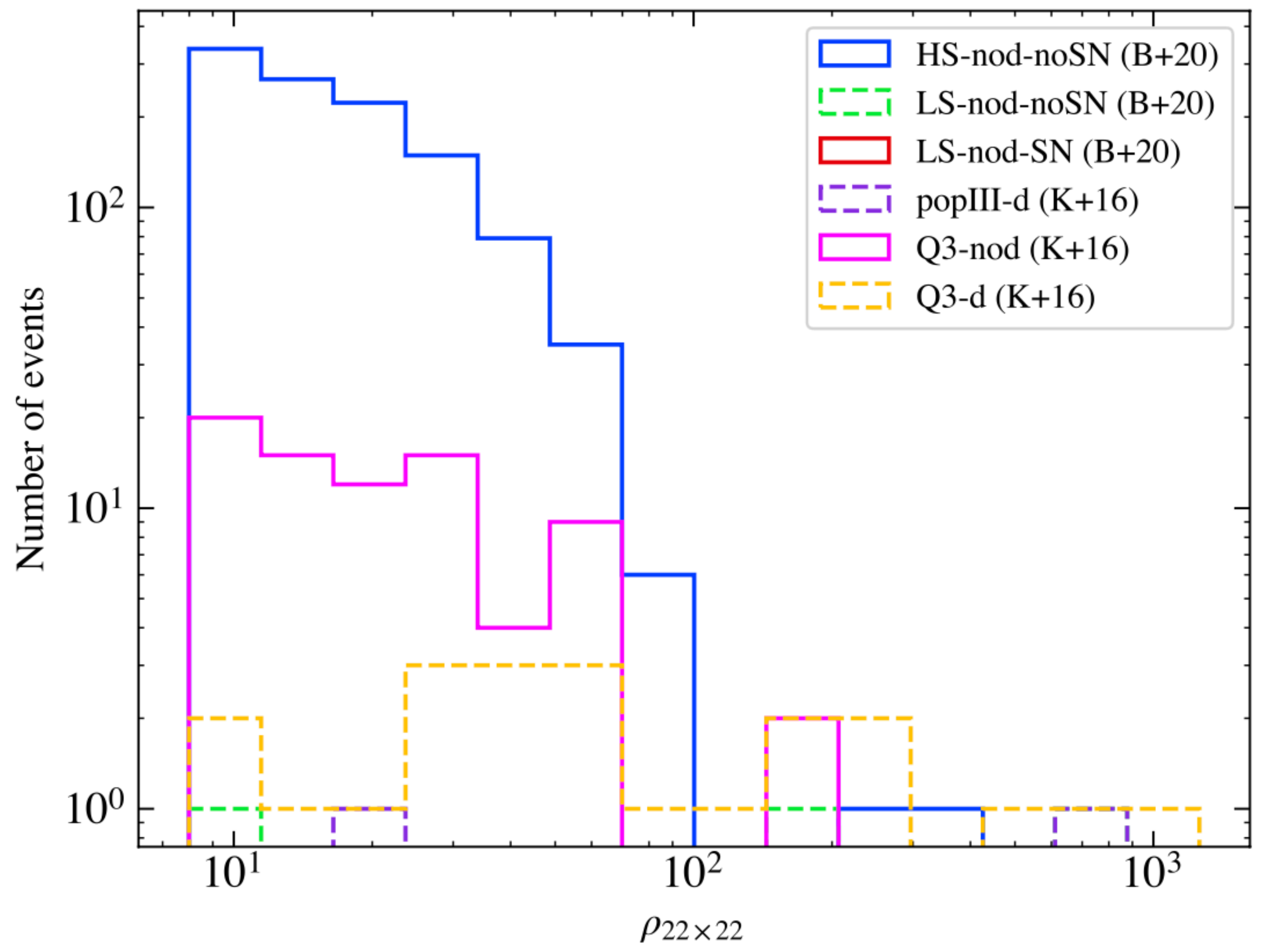
[Yi+’24]





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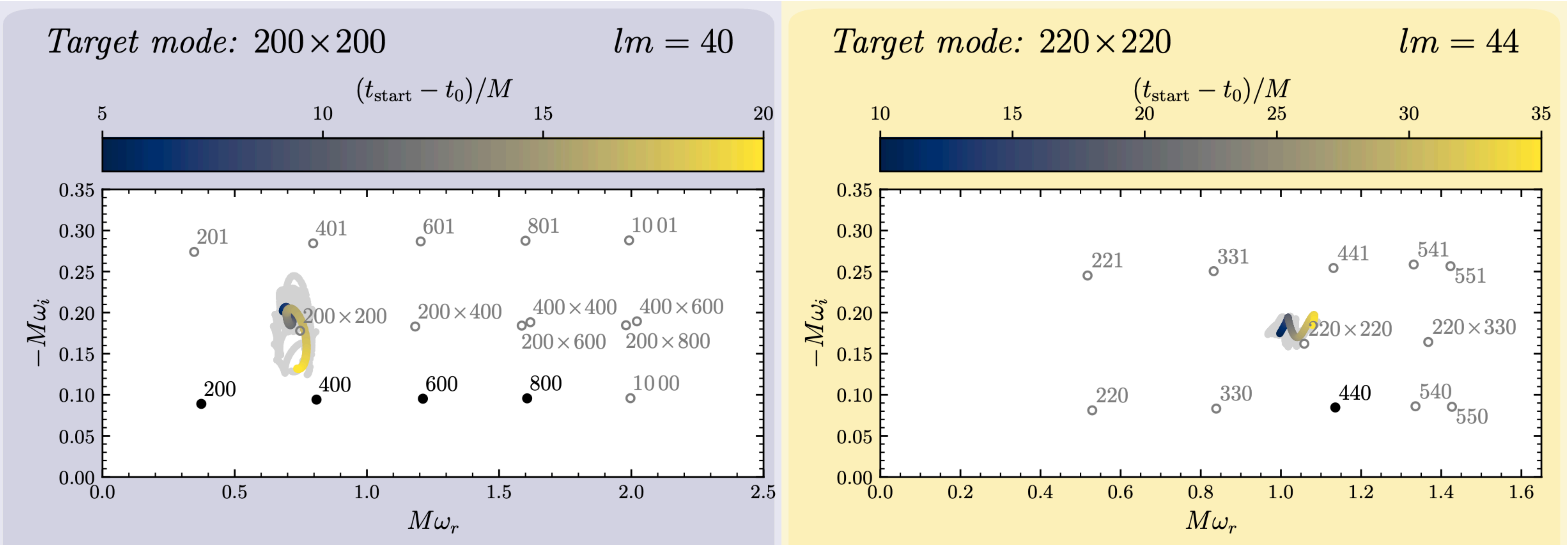
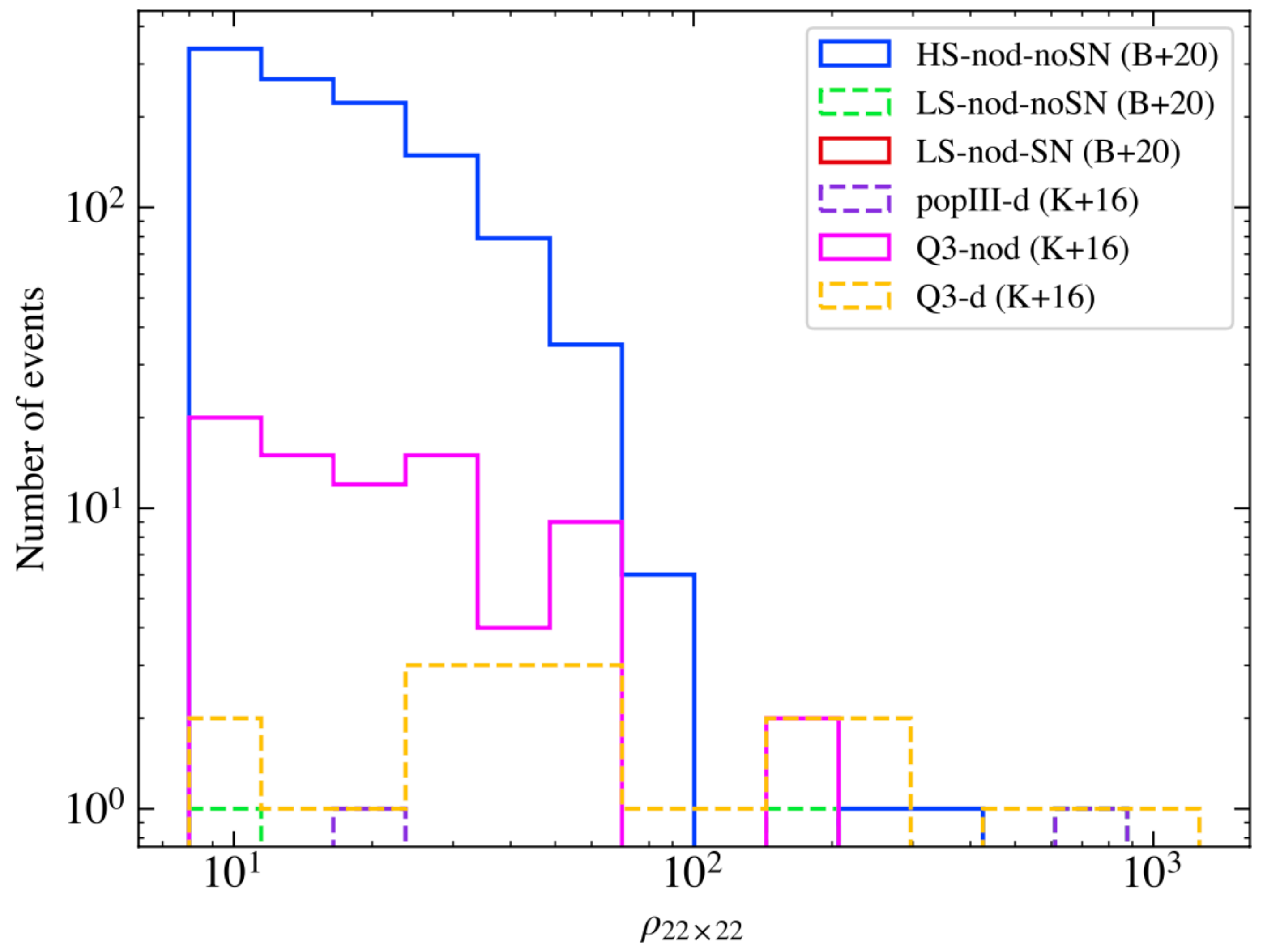
⟶ nonlinear phenomena in RD dynamics

Absorption-Induced Excitations (AIEs)

[Bamber+'21, Sberna+'22, Torres+'23, DP+'23, May+'24, Zhu+'24…]





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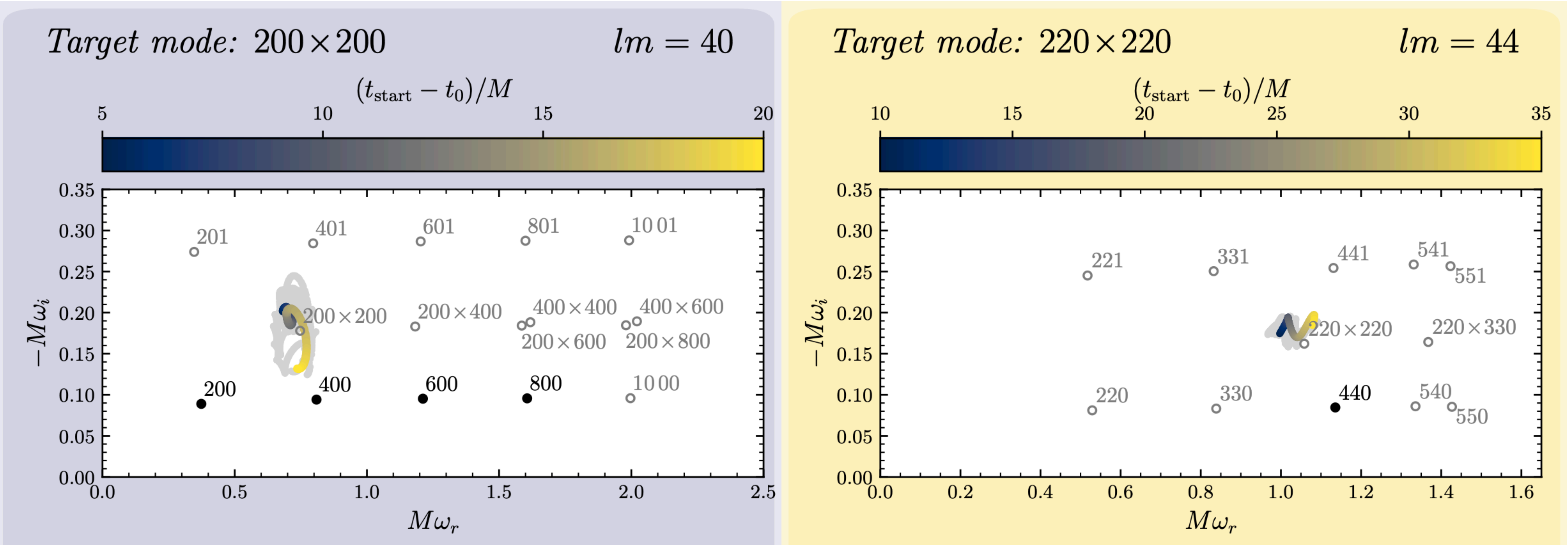
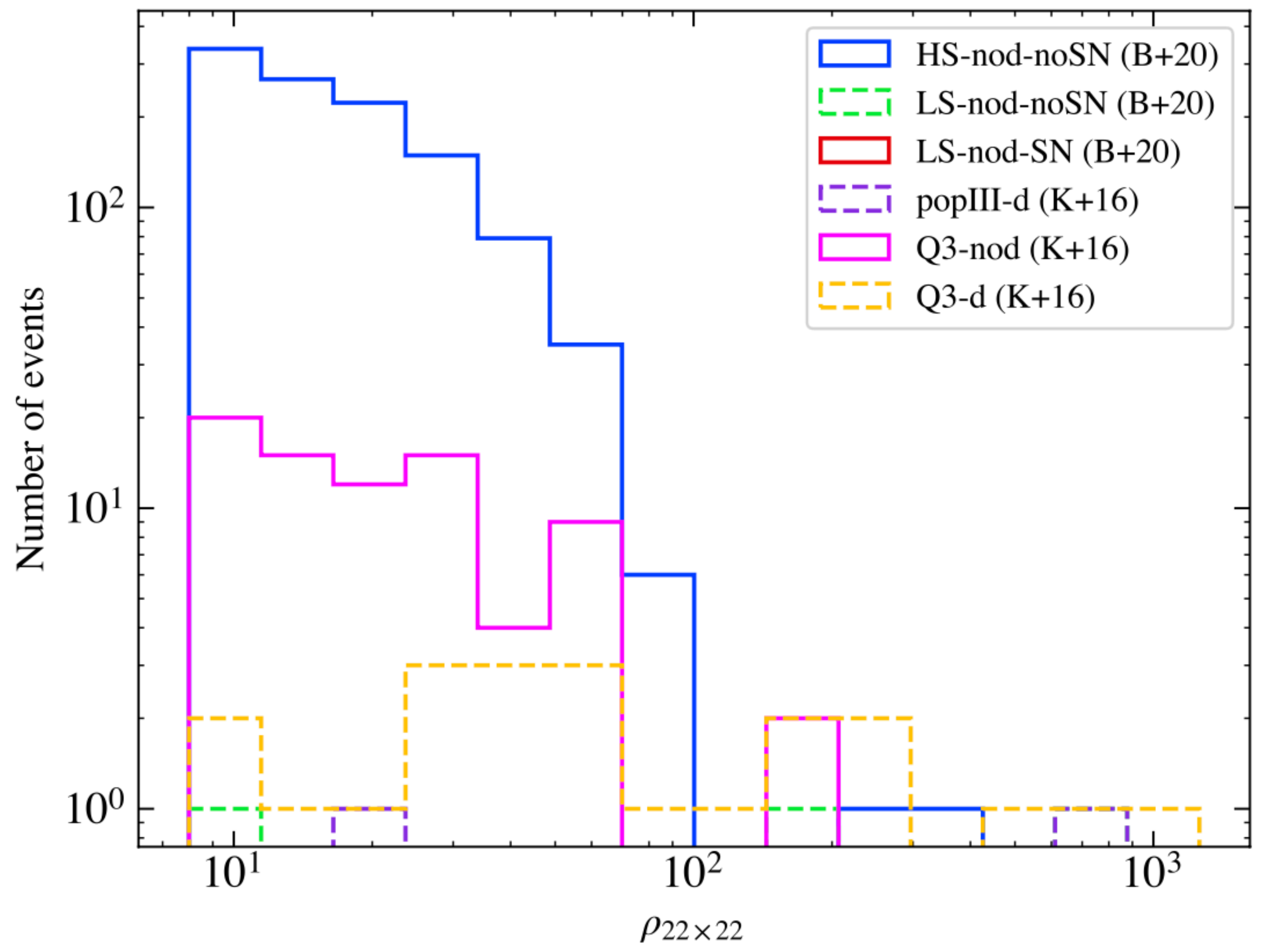
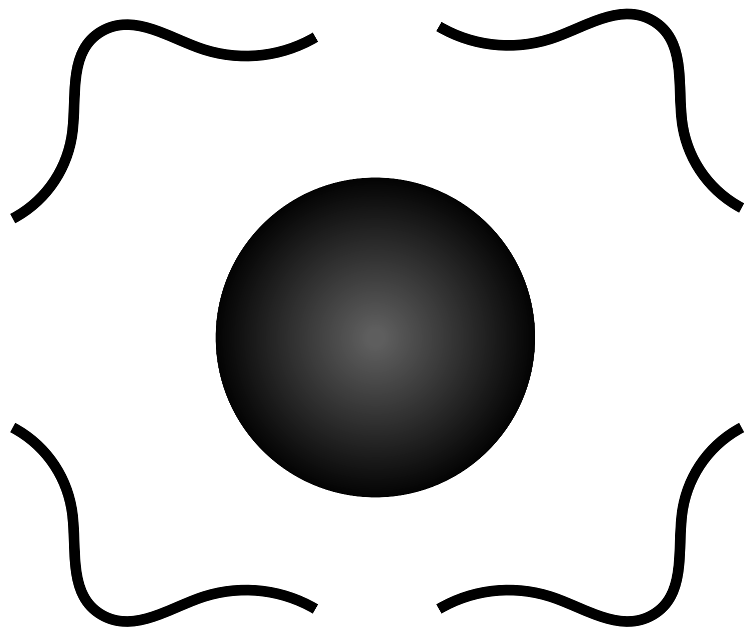
Third-order effect in perturbation theory





Higher-order Black Hole Perturbation Theory



$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(1)}_{\mu\nu}$$

$$\mathscr{L}\left[h^{(1)}\right] = 0$$

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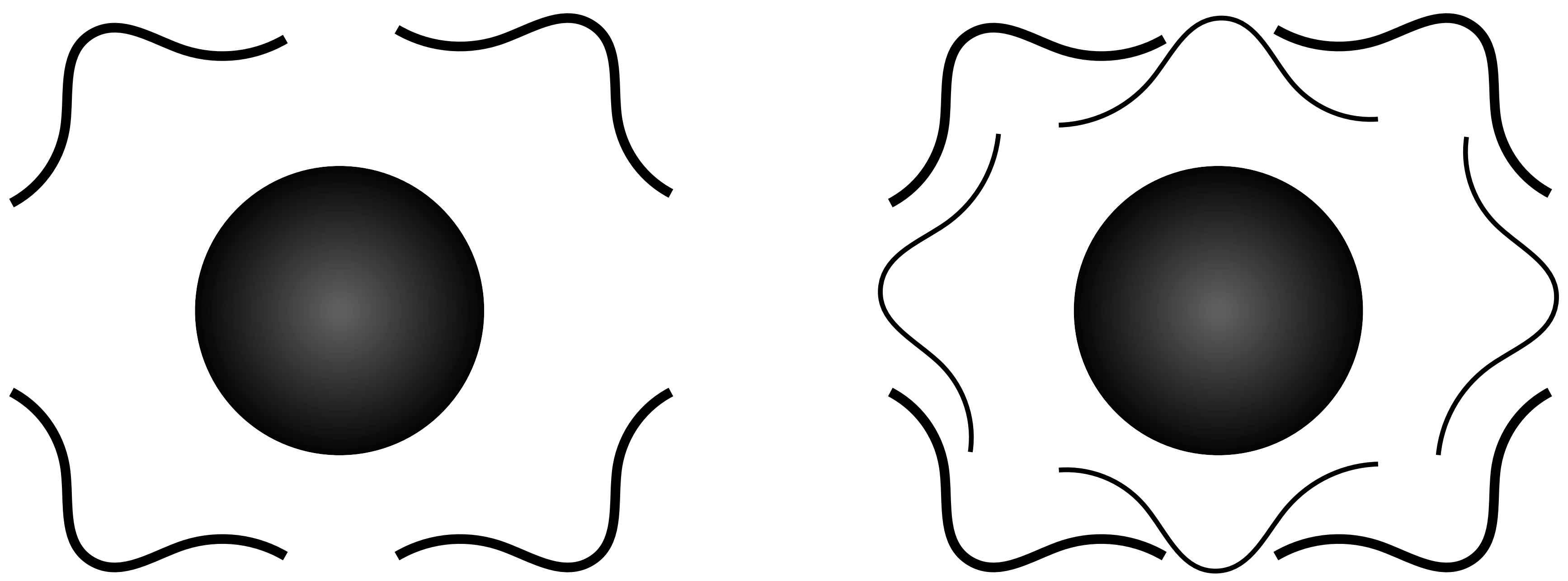
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}$$



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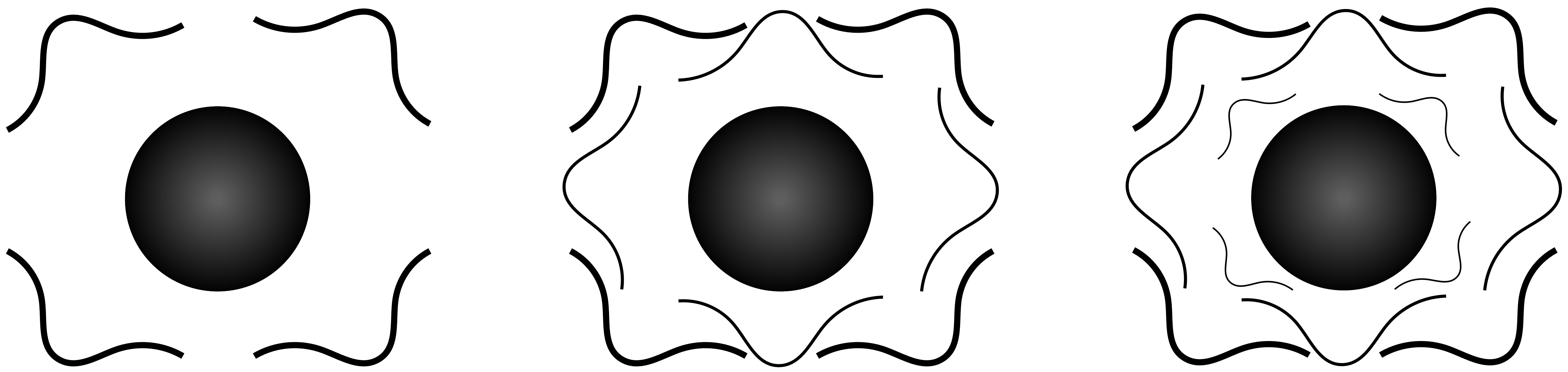

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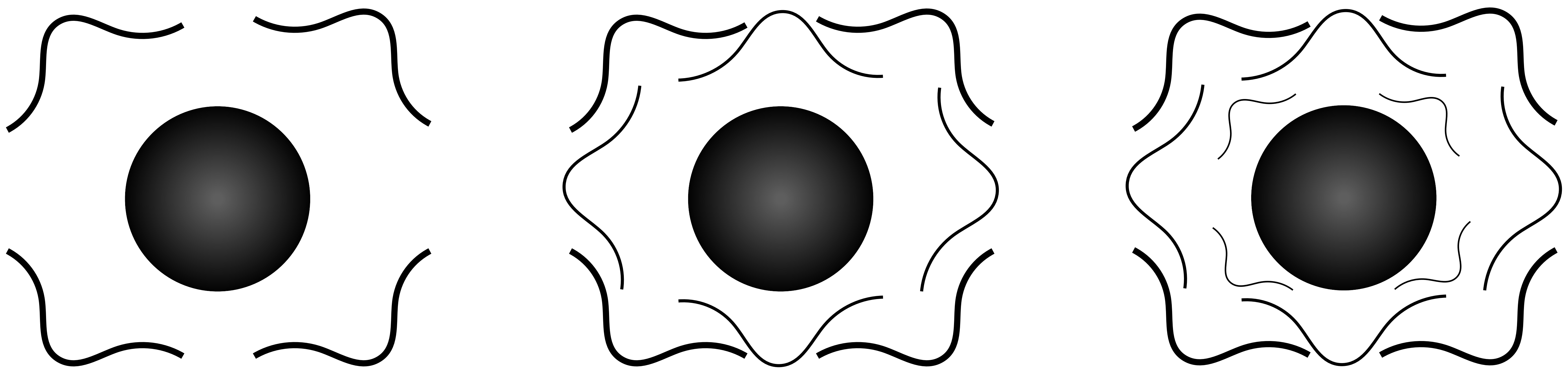

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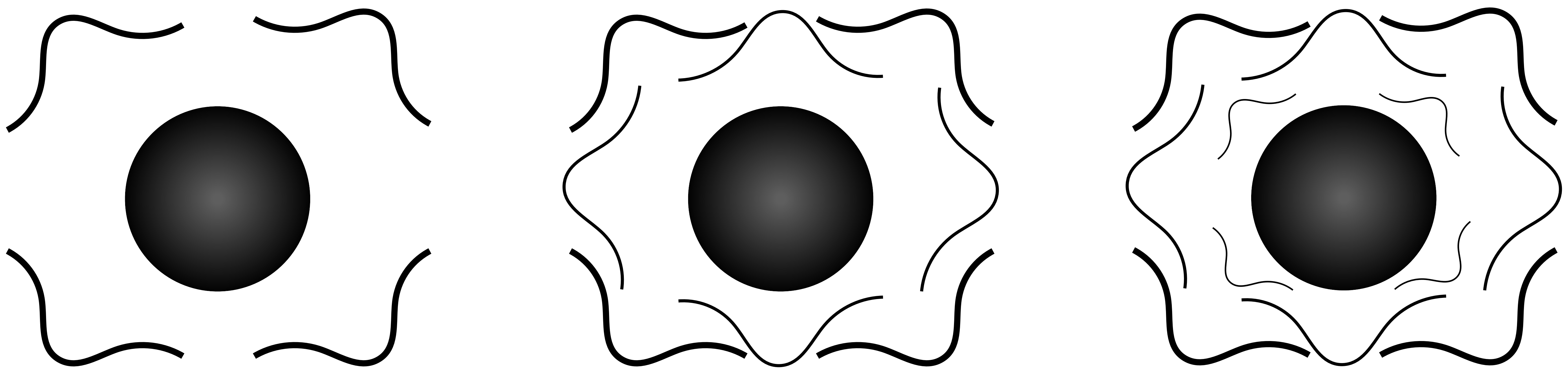

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AIEs are resonances at third-order… Long way. First steps in this talk:

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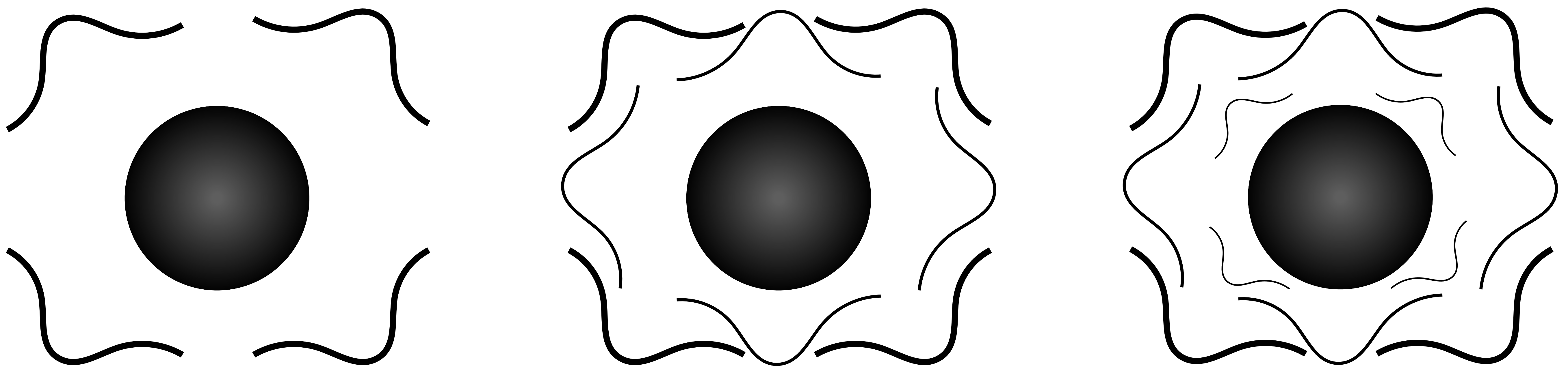

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Improvement at 1st Order	An effective approach to AIEs
A geometric approach to RWZ	*Fluctuations of dynamical BHs*
[Mukkamala and DP '24]	[Redondo-Yuste, DP and Cardoso '24]

Master Wave Equations *à la* Regge-Wheeler-Zerilli +

Spherical Symmetry: an extremely powerful assumption

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Spherical Symmetry: an extremely powerful assumption

Background

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)\Omega_{AB}dz^A dz^B$$

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Even Sector

Odd Sector

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[Regge and Wheeler '57]

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- EOMs reduce to single decoupled wave equation
  [Regge, Wheeler, Zerilli +]
- Even and odd sectors are *isospectral*
  [Chandrasekar +]

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- Non-systematic… Geometric origin?
- Artificial difference between sectors
  → need 2 propagators

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Background

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)\Omega_{AB}dz^A dz^B$$

$$h = h_{ab}(y)Y(z)dy^a dy^b + k(y)Y(z)\Omega_{AB}dz^A dz^B$$

Even Sector

$$+2j_a(y)X_A(z)dy^a dz^A$$

Odd Sector

Odd Sector

$$\begin{aligned} P_a &= -\tilde{\Box}\tilde{h}_a + \tilde{h}^b{}_{:ab} + \frac{2}{r}\left(r^{,b}\tilde{h}_{b:a} - r_{,a}\tilde{h}^b{}_{:b}\right) \\ &+ \frac{1}{r^2}\left(l(l+1) - \frac{2M}{r}\right)\tilde{h}_a - \frac{2}{r^2}r_{,a}r^{,b}\tilde{h}_b \\ P &= \tilde{h}^a{}_{:a}\,. \end{aligned}$$

[Regge and Wheeler '57]

Even Sector

$$\begin{aligned} Q_{ab} &= \tilde{p}^c{}_{(a:b)c} - \frac{1}{2}g_{ab}\tilde{p}^{cd}{}_{:cd} - \frac{1}{2}\tilde{p}^c{}_{c:ab} - \frac{1}{2}\left(\tilde{\Box}\tilde{p}_{ab} - g_{ab}\tilde{\Box}\tilde{p}^c{}_c\right) \\ &+ \frac{2}{r}r_{,c}\left(\tilde{p}^c{}_{(a:b)} - g_{ab}\tilde{p}^{cd}{}_{:d}\right) - \frac{r^{,c}}{r}\left(\tilde{p}_{ab:c} - g_{ab}\tilde{p}^d{}_{d:c}\right) + \frac{l(l+1)}{2r^2}\tilde{p}_{ab} \\ &- \frac{1}{r^2}g_{ab}r^{,c}r^{,d}\tilde{p}_{cd} - \frac{1}{2r^2}g_{ab}\left[l(l+1) + \frac{2M}{r}\right]\tilde{p}^c{}_c \\ &- \tilde{K}_{:ab} + g_{ab}\tilde{\Box}\tilde{K} - \frac{2}{r}r_{(,a}\tilde{K}_{,b)} + \frac{3}{r}g_{ab}r^{,c}\tilde{K}_{,c} - \frac{(l+2)(l-1)}{2r^2}g_{ab}\tilde{K}\,, \\ Q_a &= \tilde{p}^b{}_{a:b} - \tilde{p}^b{}_{b:a} + \frac{r_{,a}}{r}\tilde{p}^b{}_b - \tilde{K}_{,a}\,, \\ Q^\flat &= \tilde{\Box}\tilde{p}^a{}_a - \tilde{p}^{ab}{}_{:ab} - \frac{2}{r}r^{,b}\tilde{p}^a{}_{b:a} + \frac{r^{,a}}{r}\tilde{p}^b{}_{b:a} - \frac{1}{2}\frac{l(l+1)}{r^2}\tilde{p}^a{}_a + \frac{2}{r}r^{,a}\tilde{K}_{,a} + \tilde{\Box}\tilde{K} \\ Q^\sharp &= -\tilde{p}^a{}_a, \end{aligned}$$

Master Wave Equations

$$\left(\Box - V_{RW,ZM}(r)\right)\Phi_{RW,Z} = 0$$

- EOMs reduce to single decoupled wave equation [Regge, Wheeler, Zerilli +]
- Even and odd sectors are *isospectral* [Chandrasekar +]

- Non-systematic… Geometric origin?
- Artificial difference between sectors → need 2 propagators

[Lenzi and Sopuerta '21]

Master Wave Equations From Curvature Wave Equations

$$U(1) \ : \quad \mathbb{F}_{\mu\nu} \equiv F_{\mu\nu} - i \star F_{\mu\nu} \ , \quad \triangle \equiv \star\, d \star d + d \star d \star \ , \qquad \longrightarrow \qquad \boxed{\triangle\, \mathbb{F}_{\mu\nu} = 0}$$

Electromagnetic CWE

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Gravitational CWE

Expand self-dual curvature & Linearise CWEs

Applied to Teukolsky's derivation

[Ryan '74]

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Use to geometrise RWZ
[Mukkamala and DP '24]

Master Wave Equations From Curvature Wave Equations

$$ds^2 = g_{ab}(y)dy^a dy^b + r^2(y)\Omega_{AB}dz^A dz^B \quad (vacuum)$$

[Mukkamala and DP '24]

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$$\delta\mathbb{C}_{ABCD} = \mathcal{I}(y)Y(z)\ \epsilon_{AB}\epsilon_{CD}\,, \qquad \epsilon^{AB}\epsilon^{CD}\delta\left(\triangle\,\mathbb{C}_{ABCD}\right) = 0$$

$$\Psi(y) \equiv \frac{1}{r}\left(\mathcal{I}(y) - \frac{4Mk(y)}{r}\right)$$

$$\Box\,\Psi(y) + \left(\frac{6M}{r^3} - \frac{l(l+1)}{r^2}\right)\Psi(y) = 0$$

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$$Im\left(\Psi\right) = f(odd\ sector)$$

$\longrightarrow$ *Isospectrality as a corollary*

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Fluxes from master variables

$$\dot{E} = \frac{1}{64\pi}\sum_{l,m}\frac{|\dot{\Psi}^{(l,m)}|^2}{l(l+1)(l-1)(l+2)}$$

$$\dot{J} = \frac{1}{64\pi}\sum_{l,m} im\frac{\bar{\Psi}^{(l,m)}\dot{\Psi}^{(l,m)} - \Psi^{(l,m)}\dot{\bar{\Psi}}^{(l,m)}}{l(l+1)(l-1)(l+2)}$$

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- ✓ Geometric origin of mast. eqs.
- ✓ Even-Odd-symmetric
- ★ Bonus: RW eq. for even sector!

[Mukkamala and DP ’24]

Ringdown of Dynamical Spacetimes

*An effective approach to **gravitational** AIEs*

Pure radiation field:	Solution (Vaidya $m(v)$) :	Wave equation:
$G_{\mu\nu} = \Phi K_\mu K_\nu \,, \quad K^\mu K_\mu = 0$	$ds^2 = -f(v,u)dudv + r^2(u,v)d\Omega^2$	$\left[\partial^2_{uv} - \frac{f}{r}\left(\frac{3m(v)}{r^2} - \frac{l(l+1)}{2r}\right)\right]\Psi = \frac{2f}{r^2}F(v)$

Scalar and electromagnetic relaxation: [Abdalla+ '06, Chirenti+ '11]

[Redondo-Yuste, Cardoso and DP '23]

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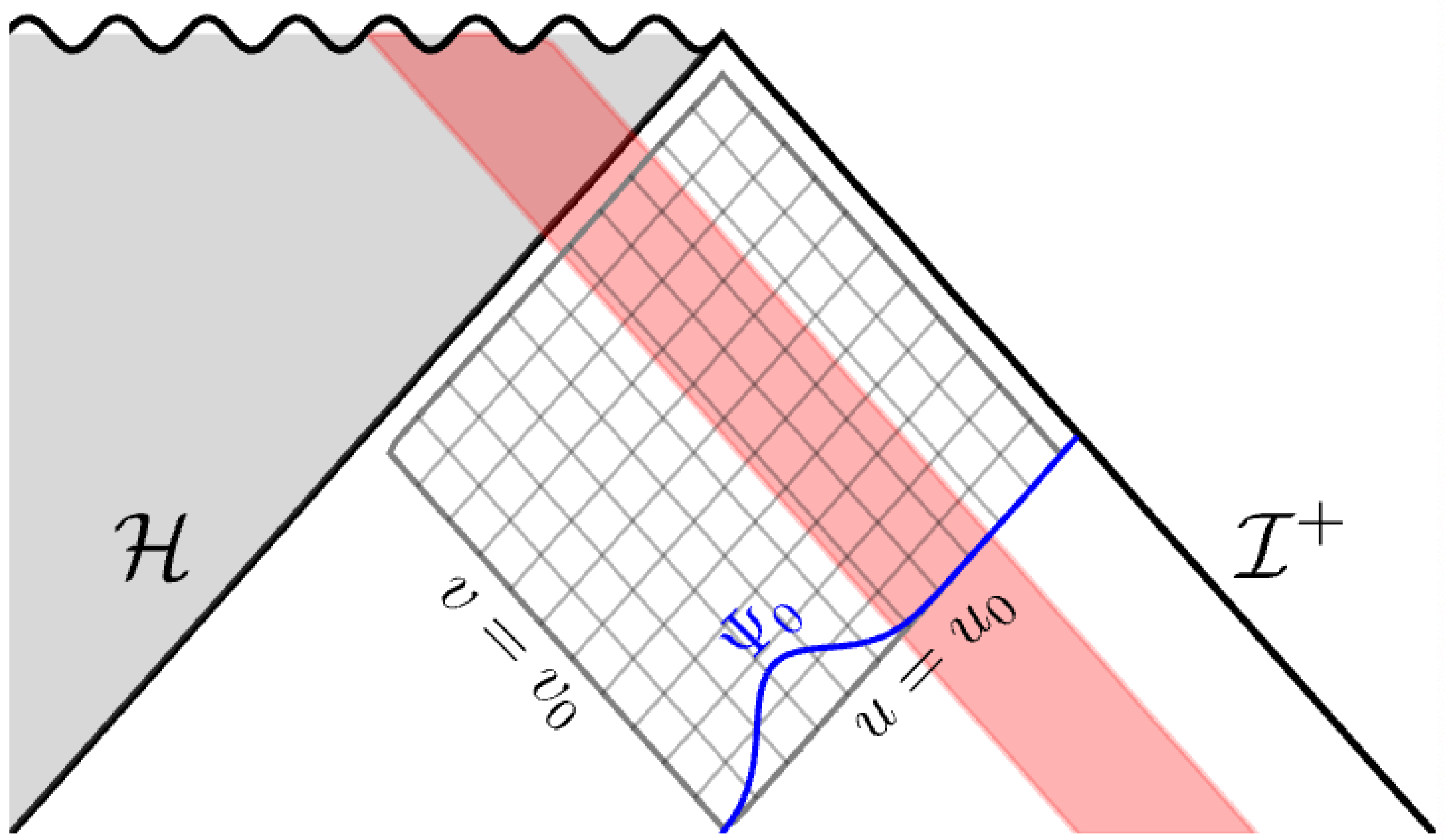

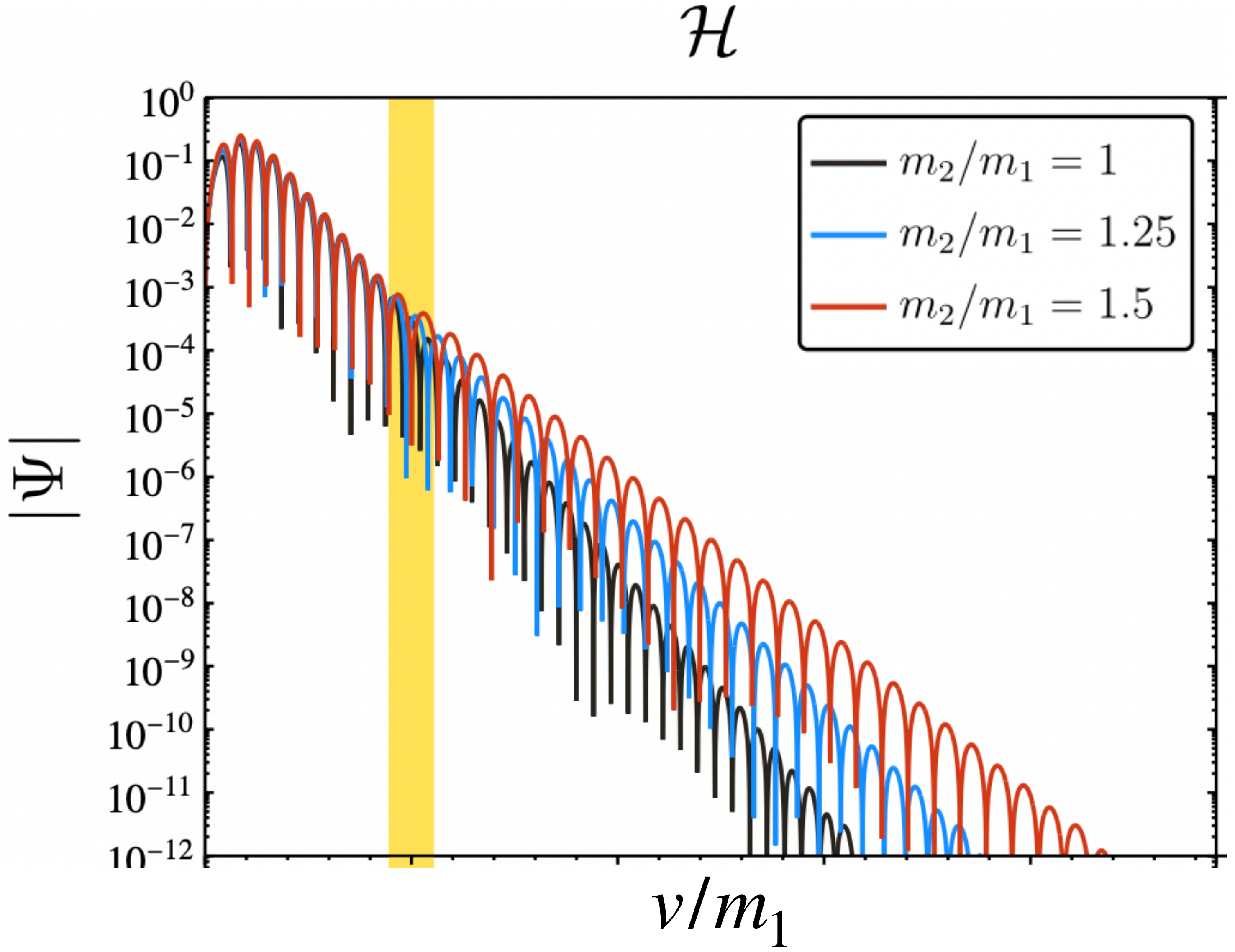

$$m(v)\omega(v) \sim constant = m_{1,2}\omega_{1,2}$$

[Redondo-Yuste, Cardoso and DP '23]

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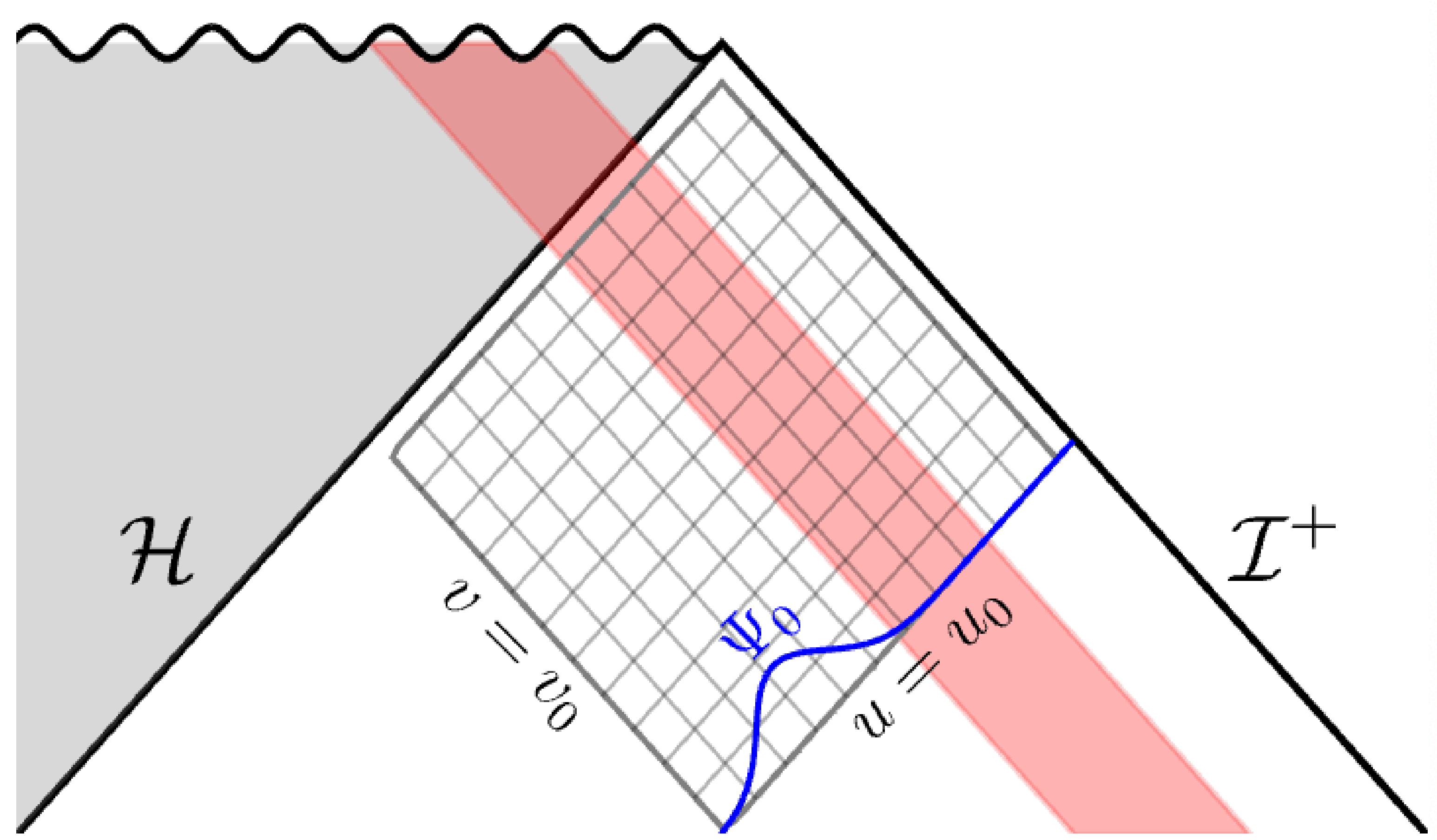

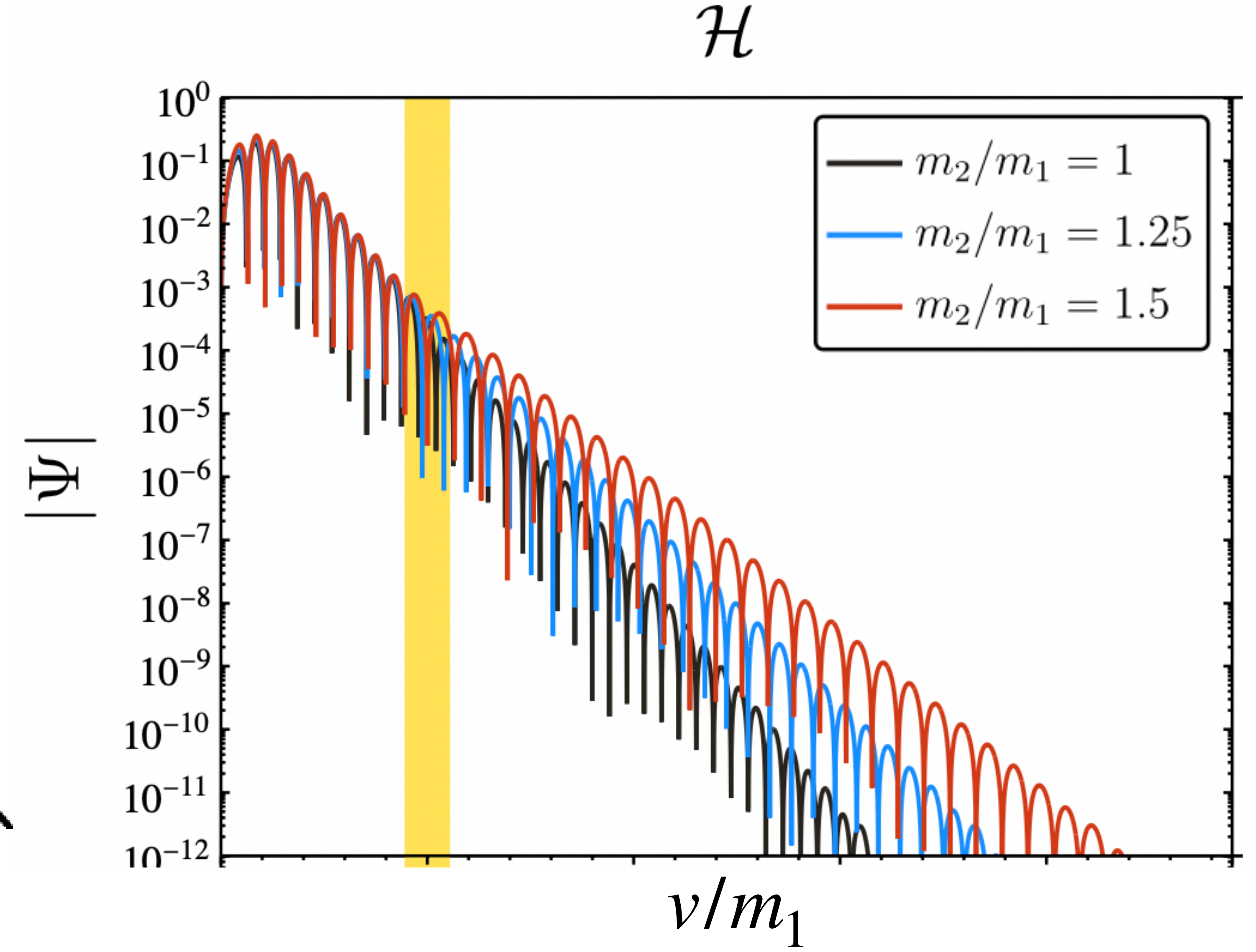

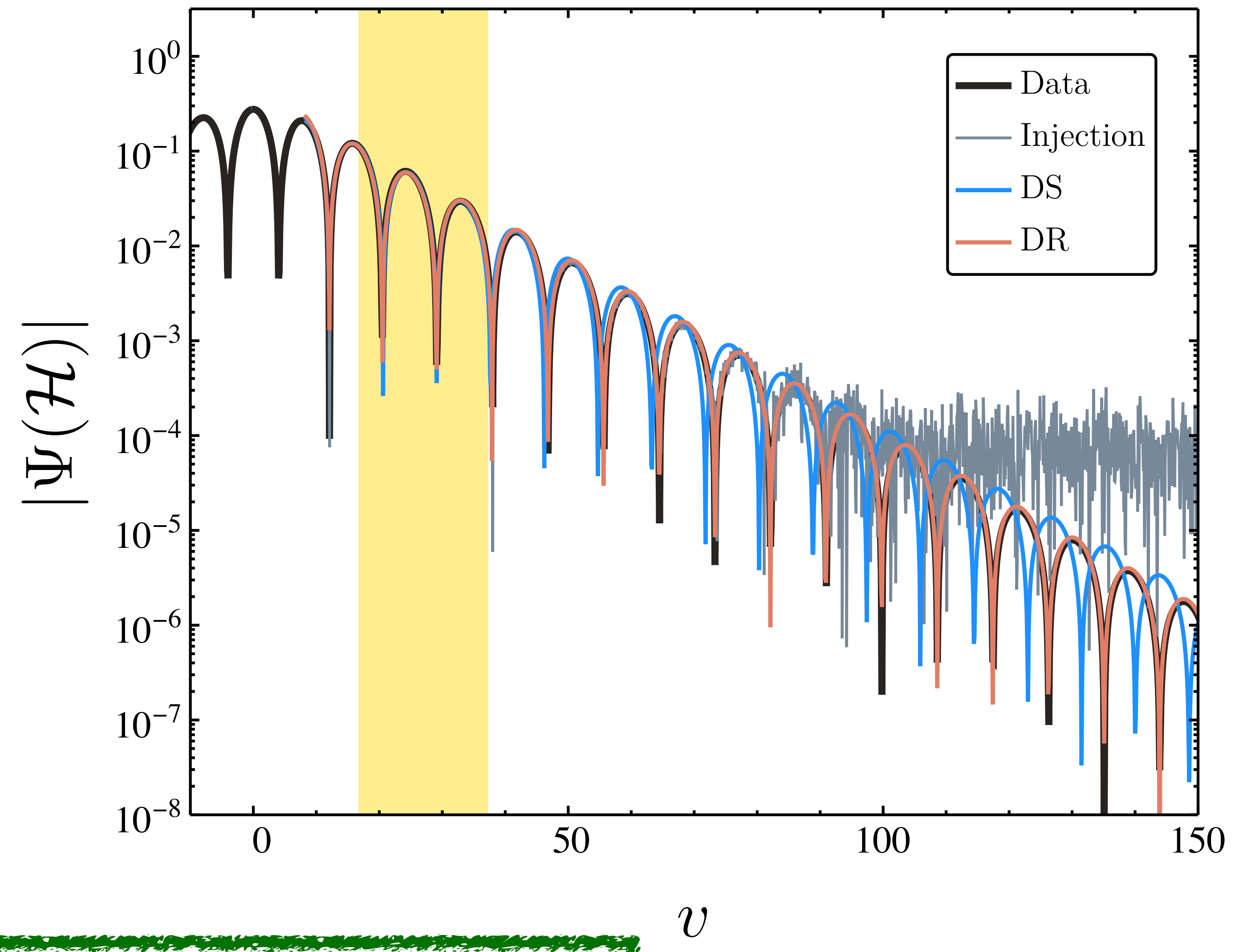

$m(v)\omega(v) \sim constant = m_{1,2}\omega_{1,2}$ ➔

$$\Psi = A(v)e^{i\omega(v)v} + c.c.\,, \quad A(v) = \tilde{A}\left[1 + \tilde{Q}\frac{\delta m(v)}{m_2 - m_1}\right], \quad \omega(v) = \frac{m_1\omega_{220}}{m(v)}\,.$$

Dynamical Ringdown

[Redondo-Yuste, Cardoso and DP '23]

Summary and future directions

- New approach to perturbation theory in spherical symmetry based on CWEs:
  - Geometric origin of master wave equations (simplifies derivation drastically)
  - Even-odd symmetric $\rightarrow$ isospectrality
  - Easily extended to matter
- Effective approach to AIEs: *gravitational* relaxation of Vaidya spacetime
  - DR: $m(v)\omega(v) = \textit{constant}$ and time-varying amplitudes
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Gràcies!