

# Unifying the inflation with dark energy in higher-derivative gravity

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## VIABLE INFLATION

Some facts about inflation are well-known:

- ▶ Inflation is described by a quasi de Sitter solution where  $R \simeq 12H^2 \sim M_{Pl}^2$ ;
- ▶ We need a graceful exit from inflation ( $\dot{H} < 0$ );
- ▶ The amount of inflation is measured by  $\mathcal{N} \equiv \log [a_f/a_i]$  and  $55 < \mathcal{N} < 65$  to solve the problem of initial conditions;
- ▶ The fluctuations of energy density at the end of inflation are  $\Delta_{\mathcal{R}}^2 (\equiv |\delta\rho/\rho|) \simeq 10^{-9}$ ;
- ▶ The Hubble flow functions

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = -\frac{2\dot{H}}{H^2} + \frac{\ddot{H}}{H\dot{H}} \equiv \frac{\dot{\epsilon}_1}{H\epsilon_1},$$

must be small during inflation. For scalar field inflation the spectral index  $n_s$  and the tensor-to-scalar ratio  $r$  are given by,

$$n_s = 1 - 2\epsilon_1 - \epsilon_2, \quad r = 16\epsilon_1,$$

and the very last Planck data constrain them as

$$n_s = 0.9603 \pm 0.0073 \text{ (68\% CL)} \text{ and } r < 0.11 \text{ (95\% CL)}.$$

## MODIFIED GRAVITY THEORIES FOR INFLATION

$$I = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \sqrt{-g} F(R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma}, R_{\mu\nu} R^{\mu\nu} \dots).$$

A simple example of modified gravity theory is given by  $f(R)$ :

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + f(R) \right],$$

where  $f(R)$  is a generic function of the Ricci scalar only. the field equations read

$$F'(R) \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \kappa^2 T_{\mu\nu}^{(\text{matter})} + \left\{ \frac{1}{2} g_{\mu\nu} [F(R) - RF'(R)] \right. \\ \left. + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) F'(R) \right\}, \quad F(R) = R + 2\kappa^2 f(R)$$

This theories can be used to study inflation.

## $F(R)$ -FRIEDMANN-LIKE EQUATIONS

On flat FRW metric the field equations read

$$\frac{3H^2}{\kappa^2} = \rho_{\text{eff}}, \quad -\frac{(3H^2 + 2\dot{H})}{\kappa^2} = p_{\text{eff}},$$

where

$$\rho_{\text{eff}} = \left[ (Rf_R - f) - 6H\dot{f}_R - 6H^2 f_R \right],$$
$$p_{\text{eff}} = \left[ (f - Rf_R) + 4H\dot{f}_R + 2\ddot{f}_R + (4\dot{H} + 6H^2)f_R \right].$$

In this case, the Hubble flow functions have to be replaced by the variables

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_3 = \frac{\kappa^2 \dot{f}_R}{H(1 + 2\kappa^2 f_R)}, \quad \epsilon_4 = \frac{\ddot{f}_R}{H\dot{f}_R}.$$

The spectral index and the tensor-to-scalar ratio are given by

$$n_s = 1 - 4\epsilon_1 + 2\epsilon_3 - 2\epsilon_4, \quad r = 48\epsilon_3^2,$$

or, in terms of the standard Hubble flow functions  $\epsilon_1, \epsilon_2$ ,

$$n_s = 1 - 2\epsilon_2, \quad r = 48\epsilon_1^2.$$

## CONFORMAL TRANSFORMATIONS IN $F(R)$ -GRAVITY

The (Jordan frame) action of a  $F(R)$ -modified gravity theory,

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{F(R)}{2\kappa^2} \right],$$

can be rewritten in the Einstein frame as

$$I = \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right),$$

where

$$\tilde{g}_{\mu\nu} = e^{-\sigma} g_{\mu\nu}, \quad \sigma = -\sqrt{\frac{3}{2\kappa^2}} \ln[F_R(R)], \quad V(\sigma) = \frac{R}{F'(R)} - \frac{F(R)}{F'(R)^2}.$$

In this way, we can study inflation from modified gravity in the scalar field representation.

## EXAMPLE: $R^2$ -MODEL FOR INFLATION

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{R + \kappa^2 R^2 / \gamma}{2\kappa^2} \right].$$

In the scalar field representation we get

$$I = \int_{\mathcal{M}} d^4x \sqrt{-\tilde{g}} \left( \frac{\tilde{R}}{2\kappa^2} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right),$$

$$V(\sigma) = \frac{\gamma}{4\kappa^2} \left( 1 - e^{\sqrt{2\kappa^2/3}\sigma} \right)^2.$$

The inflation is reproduced for large (and negative) values of  $\sigma \equiv \sigma(t)$ , such that  $V(\sigma) \simeq \gamma/(4\kappa^2)$ , and the FRW metric are

$$H^2 \simeq \frac{\gamma}{12}, \quad 3H\dot{\sigma} \simeq \left( \frac{\gamma}{\sqrt{6}\kappa^2} \right) e^{\sqrt{2\kappa^2/3}\sigma}, \quad \sigma \simeq -\sqrt{\frac{3}{2\kappa^2}} \ln \left[ \frac{\sqrt{2\gamma}}{3^{3/2}} (t_0 - t) \right].$$

Here,  $t_0$  is bounded at the beginning of the inflation and  $\gamma < M_{Pl}^2$ .

The field slowly rolls down toward the potential minimum at  $\sigma \rightarrow 0^-$ ,  $V(0) = -\gamma/(4\kappa^2) < 0$ .

The slow roll parameters read

$$\epsilon = \frac{4}{3} \frac{1}{(2 - e^{-\sqrt{2\kappa^2/3}\sigma})^2}, \quad |\eta| = \frac{4}{3} \frac{1}{|2 - e^{-\sqrt{2\kappa^2/3}\sigma}|}.$$

For  $\sigma \rightarrow -\infty$  these numbers are very small. The inflation ends when such parameters are of the order of unit, namely at  $\sigma_e \simeq -0.17\sqrt{3/(2\kappa^2)}$ . The e-folds number results to be

$$\mathcal{N} \simeq \frac{3e^{-\sqrt{2\kappa^2/3}\sigma}}{4} \Big|_{\sigma_e}^{\sigma_i} \simeq \frac{1}{4} \sqrt{\frac{2\gamma}{3}} t_0, \quad |\sigma_i| \ll |\sigma_e|.$$

For example, in order to obtain  $\mathcal{N} = 60$ , we must require  $\sigma_i \simeq -\sqrt{3/(2\kappa^2)} 4.38 \simeq 1.07 M_{Pl}$ . We also have during inflation

$$\epsilon \simeq \frac{3}{4\mathcal{N}^2}, \quad |\eta| \simeq \frac{1}{\mathcal{N}}, \quad n_s \simeq 1 - \frac{2}{\mathcal{N}}, \quad r \simeq \frac{12}{\mathcal{N}^2}.$$

Since  $1 > n_s > 1 - \sqrt{0.11/3} \simeq 0.809$  when  $r < 0.11$ , these indices are compatible with Planck data. For example, for  $\mathcal{N} = 60$ , one has  $n_s = 0.967$  and  $r = 0.003$ .



## RECONSTRUCTION OF $F(R)$ -GRAVITY FROM $V(\sigma)$

Since the relation between potential of scalar field in Einstein frame representation and modified gravity in Jordan frame is given by (\*\*)

$$V(\sigma) \equiv \frac{R}{F_R(R)} - \frac{F(R)}{F_R(R)^2} = \frac{1}{2\kappa^2} \left\{ e^{(\sqrt{2\kappa^2/3})\sigma} R \left( e^{-(\sqrt{2\kappa^2/3})\sigma} \right) - e^{2(\sqrt{2\kappa^2/3})\sigma} F \left[ R \left( e^{-(\sqrt{2\kappa^2/3})\sigma} \right) \right] \right\},$$

by taking the derivative with respect to  $R$ , we get

$$RF_R(R) = -2\kappa^2 \sqrt{\frac{3}{2\kappa^2}} \frac{d}{d\sigma} \left( \frac{V(\sigma)}{e^{2(\sqrt{2\kappa^2/3})\sigma}} \right).$$

As a result, giving the explicit form of the potential  $V(\sigma)$ , thanks to the relation  $\sigma = -\sqrt{3/2(\kappa^2)} \ln[F_R(R)]$ , we obtain an equation for  $F_R(R)$ , and therefore the  $F(R)$ -gravity model in the Jordan frame. In this process, one introduces an integration constant, which has to be fixed by requiring that (\*\*) holds true.

$$V(\sigma) \sim c_0 + c_1 \exp[\sigma] + c_2 \exp[2\sigma]: R + R^2 + \Lambda\text{-models}$$

Generalization of  $R^2$ -model to the case with cosmological constant (viable inflation):

$$V(\sigma) = \left[ c_0 + c_1 e^{\sqrt{2\kappa^2/3}\sigma} + c_2 e^{2\sqrt{2\kappa^2/3}\sigma} \right].$$

It follows

$$F_R(R) = -\frac{c_1}{2c_0} + \frac{R}{2c_0}, \quad F(R) = -\frac{c_1}{2c_0}R + \frac{R^2}{4c_0} + \Lambda$$

with  $\Lambda = c_1^2/(4c_0) - c_2$ . Thus, by putting  $-c_1/(2c_0) = 1$ ,

$$F(R) = R + \frac{R^2}{4c_0} + c_0 - c_2.$$

Interesting cases:

- ▶  $c_2 = 0$ : the model admits also the Reissner-Nordström-de Sitter solution with two free parameters.
- ▶  $c_0 = c_2$ : we recover the Starobinsky model  $F(R) = R + R^2/(4c_0)$ .

$$V(\sigma) \sim \gamma \exp[-n\sigma], n > 0: c_0 R^{\frac{n+2}{n+1}}\text{-models}$$

For the potential (\*\*)

$$V(\sigma) = \frac{\alpha}{\kappa^2} \left(1 - e^{\sqrt{2\kappa^2/3}\sigma}\right) + \frac{\gamma}{\kappa^2} e^{-n\sqrt{2\kappa^2/3}\sigma}, \quad \gamma, n > 0,$$

by putting  $\alpha = -\gamma(n+2)$ ,  $\gamma (\sim M_{Pl}^2) \gg 1$ , we obtain at the perturbative level ( $c_1 \propto \gamma^{-1}$ ,  $c_2 \propto \gamma^{-2}$ ...)

$$F_R(R \ll \gamma) \simeq 1 + c_1 R + c_2 R^2 + \dots \quad F_R(R \gg \gamma) \simeq \left(\frac{1}{4(n+2)}\right)^{\frac{1}{1+n}} \left(\frac{R}{\gamma}\right)^{\frac{1}{n+1}}.$$

Note: if in the Einstein frame inflation is realized at  $R_{EF} \sim \gamma$ , in the Jordan frame  $R_{JF} \simeq e^{-\sqrt{2\kappa^2/3}\sigma} R_{EF} \gg \gamma$ . Thus, the potential (\*\*) is realized by a model which returns GR at small curvatures and  $F(R) = c_0 R^{\frac{n+2}{n+1}}$  at high curvatures.

The potential (\*\*) has a minimum at  $\sigma \rightarrow 0^-$  when inflation ends; for  $\sigma \rightarrow -\infty$  the slow roll conditions must be satisfied.

For  $\sigma \rightarrow -\infty$ ,

$$V(\sigma) \simeq \frac{\gamma}{\kappa^2} e^{-n\sqrt{2\kappa^2/3}\sigma}, \quad H^2 \simeq \frac{\gamma}{3} e^{-n\sqrt{2\kappa^2/3}\sigma}, \quad \epsilon \simeq \frac{n^2}{3}, \quad |\eta| \simeq \frac{2n^2}{3}.$$

It means,  $0 < n \ll 1$ . Then,

$$3H\dot{\sigma} \simeq \sqrt{\frac{2}{3\kappa^2}} \gamma n e^{-n\sqrt{2\kappa^2/3}\sigma}, \quad \sigma = \frac{2}{n} \sqrt{\frac{3}{2\kappa^2}} \ln \left[ \frac{n^2}{3} \sqrt{\frac{\gamma}{3}} (t_0 + t) \right],$$

and

$$H = \frac{3}{n^2(t_0 + t)}, \quad \frac{\ddot{a}}{a} = H^2 + \dot{H} = \frac{3}{n^2(t_0 + t)^2} \left( \frac{3}{n^2} - 1 \right) > 0, \quad (n < \sqrt{3}).$$

We have an acceleration as soon as  $\epsilon < 1$ .

$$N \simeq -\frac{1}{n} \sqrt{\frac{3\kappa^2}{2}} \sigma \Big|_{\sigma_e}^{\sigma_i} \simeq -\frac{3}{n^2} \ln \left[ \frac{n^2}{3} \sqrt{\frac{\gamma}{3}} t_0 \right], \quad n_s \simeq 1 - \frac{2n^2}{3}, \quad r \simeq \frac{16n^2}{3}.$$

The result suggests that only the models close to  $R^2$ -gravity are able to produce this kind of inflation, since  $n$  must be extremely close to  $0^+$  (we remember  $F(R) = c_0 R^{\frac{n+2}{n+1}}$ ).

## OTHER EXAMPLES

- ▶ From the potential

$$V(\sigma) = \frac{3\gamma}{4\kappa^2} - \frac{\gamma}{\kappa^2} e^{\sqrt{2\kappa^2/3}\sigma/2}, \quad 1 \ll \gamma \sim M_{Pl}^2,$$

we get

$$F(R) = \frac{R}{2} + \frac{R^2}{6\gamma} + \frac{\sqrt{3}}{36} (4R/\gamma + 3)^{3/2} + \frac{\gamma}{4}.$$

Note that  $F(R \ll \gamma) \simeq R + \gamma/2$ . We can set the cosmological constant equal to zero, adding a suitable term in the potential proportional to  $\exp\left[2\sqrt{2\kappa^2/3}\sigma\right]$ , which does not change the dynamics of inflation. The model reproduces a viable inflation with

$$\epsilon \simeq \frac{3}{N^2}, \quad |\eta| \simeq \frac{1}{N}, \quad n_s \simeq 1 - \frac{1}{N}, \quad r \simeq \frac{48}{N^2}.$$

- ▶ From the potential (\*\*)

$$V(\sigma) = \frac{\gamma(2-n)}{2\kappa^2} - \frac{\gamma}{\kappa^2} e^{n\sqrt{2\kappa^2/3}\sigma}, \quad 1 \ll \gamma \sim M_{Pl}^2, \quad 0 < n < 2,$$

we get ( $c_1 \propto \gamma^{-1}$ ,  $c_2 \propto \gamma^{-2}$ ,  $c_3 \propto \gamma^{-3}$ ...)

$$F_R(R \ll \gamma) \simeq 1 + c_1 R + c_2 R^2 + c_3 R^3 + \dots$$


$$F_R(R \gg \gamma) \simeq \left( \frac{R}{2\gamma(2-n)} \right) + \left( \frac{R}{2\gamma(2-n)} \right)^{1-n},$$

such that models like  $F(R) = b_1 R^2 + b_2 R^{2-n}$  realize the potential (\*\*) at high curvatures. The model reproduces a viable inflation with

$$\epsilon \simeq \frac{3}{4n^2 N^2}, \quad |\eta| \simeq \frac{1}{N}, \quad n_s \simeq 1 - \frac{1}{N}, \quad r \simeq \frac{12}{n^2 N^2}.$$

Since  $1 > n_s > 1 - (\sqrt{0.11/12})n \simeq 1 - (0.0957)n$  when  $r < 0.11$ , one has that

$$0.3386 < n < 1,$$

in order to make the spectral indexes compatible with the Planck data (i.e.  $F(R \gg M_{Pl}) \simeq c_1 R^2 + c_2 R^\zeta$ ,  $1 \ll \zeta \leq 5/3$ ). 

## CONFORMAL ANOMALY

The renormalized procedure of the matter field lagrangian at the Planck scale leads to the so-called conformal anomaly,

$$\langle T_{\mu}^{\mu} \rangle = \alpha \left( W + \frac{2}{3} \square R \right) - \beta \mathcal{G} + \xi \square R,$$

$W = C^{\xi\sigma\mu\nu} C_{\xi\sigma\mu\nu}$  being the 'square' of the Weyl tensor and  $\mathcal{G}$  the Gauss-Bonnet topological invariant. Moreover,  $\alpha$ ,  $\beta$  and  $\xi$  are related to the number of conformal fields present in the theory. For  $\mathcal{N}_{\text{super}} = 4$  SU(N) super Yang-Mills (SYM) theory,

$$\alpha = \beta = \frac{N^2}{64\pi^2}, \quad \xi = -\frac{N^2}{96\pi^2}, \quad N \gg 1$$

and  $\frac{2}{3}\alpha + \xi = 0$ , but in principle the contribution of the  $\square R$  term could be reintroduced via  $R^2$ -term in the action, such that we redefine

$$\langle T_{\mu}^{\mu} \rangle = (\alpha W - \beta \mathcal{G} + \delta \square R), \quad 0 < \alpha, \beta, \quad \delta < 0.$$

## CONFORMAL ANOMALY IN $F(R)$ -GRAVITY

The action reads

$$I = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4x \sqrt{-g} [R + 2\kappa^2 \tilde{\gamma} R^2 + f(R) + 2\kappa^2 \mathcal{L}_{\text{QC}}], \quad \tilde{\gamma} \equiv \frac{\gamma N^2}{192\pi^2} > 0.$$

The trace of field equations is ( $\alpha, \beta > 0$ )

$$R = -\kappa^2 (\alpha W - \beta G + \delta \square R) - 2f(R) + Rf_R(R) + 3\square f_R(R), \quad \delta \equiv -12\tilde{\gamma} < 0.$$

On FRW metric we must have

$$\frac{3}{\kappa^2} H^2 = \rho_{\text{QC}} + \rho_{\text{MG}} \equiv \rho_{\text{eff}}, \quad -\frac{1}{\kappa^2} (2\dot{H} + 3H^2) = p_{\text{QC}} + p_{\text{MG}} \equiv p_{\text{eff}},$$

with the continuity equation

$$\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0.$$

Note that on FRW metric  $W = 0$ .



On FRW metric the trace of field equations leads to an expression for  $\rho_{\text{eff}}$ ,

$$-\rho_{\text{eff}} + 3p_{\text{eff}} = -\frac{R}{\kappa^2} \equiv -\beta G + \delta \square R + \frac{1}{\kappa^2} (2f(R) - Rf_R(R) - 3\square f_R(R)) .$$

Now we can eliminate the pressure in the conservation law, such that

$$\begin{aligned} \frac{d}{dt} (\rho_{\text{eff}} a^4) &= -\dot{a} a^3 (-\rho_{\text{eff}} + 3p_{\text{eff}}) = \\ &= a^3 \dot{a} \left[ 24\beta \frac{\dot{a}^2 \ddot{a}}{a^3} + \delta (\ddot{R} + 3H\dot{R}) \right] - \frac{\dot{a} a^3}{\kappa^2} (2f(R) - Rf_R(R) - 3\square f_R(R)) , \end{aligned}$$

whose integration leads to

$$\rho_{\text{eff}} = \frac{\rho_0}{a^4} + 6\beta H^4 + \delta (18H^2 \dot{H} + 6\ddot{H}H - 3\dot{H}^2) + \rho_{MG} ,$$

where

$$\rho_{MG} = \frac{1}{2\kappa^2} (Rf_R(R) - f(R) - 6H^2 f_R(R) - 6H\dot{f}_R(R)) .$$

Given  $\rho_{\text{eff}}$ , we also have  $p_{\text{eff}}$  from the continuity equation. We will put  $\rho_0 = 0$ .

## APPLICATION TO INFLATION

$$f(R) = -2\Lambda_{\text{eff}} \left[ 1 - \exp\left(-\frac{R}{R_0}\right) \right], \quad R_0, \Lambda_{\text{eff}} > 0,$$

namely  $f(R) \simeq -2\Lambda_{\text{eff}}\theta(R - R_0)$ . At high curvatures  $R_0 \ll R$ , from the first Friedmann-like equation, by taking into account the conformal anomaly, we get

$$\frac{3}{\kappa^2} H^2 \simeq 6\beta H^4 + \delta \left( 18H^2 \dot{H} + 6\ddot{H}H - 3\dot{H}^2 \right) + \frac{\Lambda_{\text{eff}}}{\kappa^2},$$

such that

$$H_{\text{dS}\pm}^2 = \frac{1}{4\beta\kappa^2} \left[ 1 \pm \sqrt{1 - \frac{8\Lambda_{\text{eff}}\beta\kappa^2}{3}} \right],$$

By perturbing the equation above, it is possible to show that the only (real) unstable de Sitter solution able to describe the inflation is  $H_+$  with  $R_0 < H_+^2$ , such that we require also  $H_-^2 < R$ . The De Sitter solution is determined by the  $\mathcal{G}$ -contribution ( $\beta$  coefficient). Stability/instability depends on  $\delta < 0$  ( $R^2$  contribution to conformal anomaly).

The dynamics of inflation depends on modified gravity. By perturbing the EOMs with respect to  $R_{\text{dS}} \equiv 12H_{\text{dS}+}^2$ , one has for exponential model,

$$-\delta \left( \Delta \ddot{H}(t) + 3H_{\text{dS}} \Delta \dot{H}(t) \right) + \Delta H(t) \left( \frac{1}{\kappa^2} - 4H_{\text{dS}}^2 \beta \right) = -\frac{e^{-R_{\text{dS}}/R_0} \Lambda_{\text{eff}}}{12H_{\text{dS}} \kappa^2} \left( \frac{R_{\text{dS}}}{R_0} + 2 \right).$$

with the solution

$$\Delta H(t) = A_0 e^{\lambda t} - \frac{e^{-R_{\text{dS}}/R_0} \Lambda_{\text{eff}}}{12H_{\text{dS}} \kappa^2} \left( \frac{R_{\text{dS}}}{R_0} + 2 \right) \left( \frac{1}{\kappa^2} - 4H_{\text{dS}}^2 \beta \right)^{-1},$$

$$\lambda_{1,2} = \frac{-3H_{\text{dS}} \pm \sqrt{9H_{\text{dS}}^2 + \frac{4}{\delta} \left( \frac{1}{\kappa^2} - 4H_{\text{dS}}^2 \beta \right)}}{2},$$

where  $\lambda_1$  is real and positive for our unstable de Sitter solution and  $A_0$  is determined by putting  $\Delta H(t=0)=0$  at the beginning of inflation. One finds  $A_0 < 0$ , such that  $R$  slowly decreases during inflation.

## $\mathcal{N}$ -FOLDS, EoS PARAMETER AND SPECTRAL INDEX

The  $\mathcal{N}$ -folds follows from the fact  $H_{\text{dS}}|\Delta H(t)| \simeq H_{\text{dS}}^2$  at the end of inflation. In our case,

$$\mathcal{N} = \frac{2b}{3} \left[ -1 + \sqrt{1 + \frac{4}{9\gamma} \left( \frac{\sqrt{1 - \frac{8}{3}\zeta}}{1 + \sqrt{1 - \frac{8}{3}\zeta}} \right)} \right]^{-1}, \quad \Lambda_{\text{eff}} = \frac{\zeta}{\beta\kappa^2},$$

where  $\gamma$  comes from the  $R^2$ -term in the action (or  $\square R$  of conformal anomaly). Here, we have put  $R_{\text{dS}} = bR_0$  in the model  $f(R) = -2\Lambda_{\text{eff}}(1 - e^{-R/R_0})$  under the condition

$$1 \ll b \leq \frac{1 + \sqrt{1 - \frac{8}{3}\zeta}}{1 - \sqrt{1 - \frac{8}{3}\zeta}}.$$

As an example, for  $\Lambda_{\text{eff}} = 1/(8\beta\kappa^2)$  and  $b = 3$ , this condition is satisfied and only the unstable de Sitter solution  $R_{\text{dS}} \equiv R_{\text{dS}+}$  can be realized. In this case, to obtain  $\mathcal{N} > 76$ ,  $\gamma$  has to meet the relation  $\gamma > 3.8$ .

By taking into account the perturbations on the de Sitter solution, one has that during the inflation the effective EoS-parameter reads

$$-1 < w_{\text{eff}} \left( \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} \right) < -1 + \frac{2R_{\text{dS}}}{3R_0\mathcal{N}},$$

such that the universe is in a quintessence phase ( $R$  slowly decreases).

The slow roll parameters and spectral index are given by

$$\epsilon \simeq \frac{b^2 e^{-b\zeta} (b+2)}{\mathcal{N}^2 \left(1 - \frac{8}{3}\zeta\right)} \ll 1, \quad |\eta| \simeq \left| -\frac{b}{2\mathcal{N}} \right| \ll 1,$$
$$n_s = 1 - \frac{b}{\mathcal{N}} - \frac{6b^2 e^{-b\zeta} (b+2)}{\mathcal{N}^2 \left(1 - \frac{8}{3}\zeta\right)}, \quad r = \frac{16b^2 e^{-b\zeta} (b+2)}{\mathcal{N}^2 \left(1 - \frac{8}{3}\zeta\right)}.$$

This expressions may also satisfy the last BICEP2 results, where  $r = 0.20_{-0.05}^{+0.07}$  (68% CL). For example, if  $\mathcal{N} = 76$ , for  $b = 2, 3$  and  $4$ , we acquire  $r = 0.22, 0.23$ , and  $0.18$ , respectively.

## END OF INFLATION AND GRAVITATIONAL COUPLING

In the limit  $R/R_0 \ll 1$ , the model is  $f(R) = 2\Lambda_{\text{eff}}(1 - e^{-R/R_0}) \simeq 0$ . By perturbing the EOMs we find

$$\Delta H(t) = c_0 \cos(B_0 t)^2, \quad B_0 = \frac{1}{2} \sqrt{\frac{R_0(R_0 - 2\Lambda_{\text{eff}})}{6\Lambda_{\text{eff}} - \delta R_0^2 \kappa^2}},$$

such that, since  $\delta < 0$ , we must require  $2\Lambda_{\text{eff}} < R_0 \equiv R_{\text{dS}}/b$ .

The effective gravitational coupling  $G_{\text{eff}} = G_N / (1 + f_R(R))$ , has to be positive to correctly describe the interactions between matter and gravity and in particular  $G_{\text{eff}} \simeq G_N$  when  $R/R_0 \ll 1$ . This is true if  $2\Lambda_{\text{eff}} \ll R_0$ .

In general, the smaller the effective cosmological constant of the exponential gravity model is, the better all the viability conditions of inflation are met. In this way, our model tends to the original Starobinsky model, which takes into account the conformal anomaly only (Starobinsky, 1980). However, we have shown how the presence of a modification of gravity can realize the perturbations to drive a viable inflation.

## HIGHER DERIVATIVE QUANTUM GRAVITY

The starting action of higher-derivative gravity has the following form,

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left( \frac{R}{\kappa^2} - \Lambda + a R_{\mu\nu} R^{\mu\nu} + b R^2 + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} + d \square R \right),$$

where  $a, b, c, d$  are running coupling constants. If we introduce the Gauss Bonnet  $G$  and the “square” of the Weyl tensor  $W$  such that  $R_{\mu\nu} R^{\mu\nu} = \frac{C^2}{2} - \frac{G}{2} + \frac{R^2}{3}$  and  $R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} = 2C^2 - G + \frac{R^2}{3}$ , we obtain

$$I = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[ \frac{R}{\kappa^2(t')} - \frac{\omega(t')}{3\lambda(t')} R^2 + \frac{1}{\lambda(t')} C^2 - \gamma(t') G + \zeta(t') \square R - \Lambda(t') \right],$$

where  $t' = (t'_0/2) \log [R/R_0]^2$ ,  $t_0$  is a number and  $R_0 = 4\Lambda$  is the curvature of today universe,  $\Lambda$  being the cosmological constant.

The one-loop running coupling constants  $\lambda(t')$ ,  $\omega(t')$ ,  $\kappa^2(t')$ ,  $\Lambda(t')$ ,  $\gamma(t')$  and  $\zeta(t')$  are found from higher-derivative quantum gravity. They can be written as

$$\lambda(t') = \frac{\lambda(0)}{(1 + \lambda(0)\beta_2 t')}, \quad \omega(t') = \omega_1,$$

$$\kappa^2(t') = \kappa_0^2(1 + \lambda(0)\beta_2 t')^{0.77}, \quad \Lambda(t') = \frac{\Lambda_0}{(1 + \lambda(0)\beta_2 t')^{0.55}},$$

with  $\beta_2 = 133/10$ ,  $\omega_1 = -0.02$ ,  $\kappa_0^2 = 16\pi/M_{Pl}^2$ ,  $\Lambda_0 = 2\Lambda$ . The expressions for  $\omega(t')$ ,  $\kappa^2(t')$  and  $\Lambda(t')$  are derived by investigating the asymptotic behaviour of the running constants at high curvature. However, the derivatives of the coupling constants obey to a set of renormalization group equations. Here,  $\lambda(0)$  is a number related to the bound of inflation (large curvature), such that  $\lambda(t' = 0) = \lambda(0)$ . The form of  $\gamma(t')$  and  $\zeta(t')$ , namely the coefficients in front of  $G$  and  $\square R$ , is given by

$$\gamma(t') = \gamma_0(1 + c_1 t'), \quad \zeta(t') = \zeta_0(1 + c_2 t'), \quad c_1 \gamma_0 < 0,$$

$\gamma_0$ ,  $\zeta_0$  and  $c_{1,2}$  constants. Condition  $c_1 \gamma_0 < 0$  will be necessary to reproduce Planck data.



The model posses a de Sitter solution at

$$H_{\text{dS}}^2 \kappa_0^2 \simeq \frac{322.762}{(22085.2 - 34725.2(d\gamma/dt')) t'_0(\lambda(0)t')^{0.77}},$$

where  $(d\gamma/dt') < 0$ , and the solution is always unstable for inflationary scenario, but, what it is more important, it is able to reproduce the spectral index and the tensor-to-scalar ratio of the Planck data thanks to the contribution of the Gauss Bonnet throught the coefficient  $\gamma_0$ . Given the running coupling constant in front of the GB  $\gamma(t') = \gamma_0(1 + c_1 t')$ , one has that the model reproduces the correct spectral index (tensor-to-scalar ratio is very close to zero) when

$$-4.61 < \gamma_0 c_1 < -2.98.$$

For example, for  $c_1 = 1$  and  $\gamma_0 = -3$  we find  $n_s \simeq 0.96740$ .

## VIABLE INFLATION IN MODIFIED GRAVITY

- ▶ Models  $R/(2\kappa^2) + \gamma R^n + \text{const}$  can reproduce inflation only if  $n$  is close to  $n = 2$ .
- ▶ Other interesting extensions may be constructed by starting from quantum effects: for example, we can study inflation in the presence of trace anomaly,

$$\langle T_{\mu}^{\mu} \rangle = \alpha \left( W + \frac{2}{3} \square R \right) - \beta \mathcal{G} + \xi \square R,$$

$W, \mathcal{G}$  being the “square” of the Weyl tensor and the Gauss-Bonnet topological invariant in four dimensions. An other interesting application is the inflation from higher derivative quantum gravity in the form

$$\mathcal{L} = \sqrt{-g} \left( \frac{R}{\kappa^2} - \Lambda + a R_{\mu\nu} R^{\mu\nu} + b R^2 + c R_{\mu\nu\xi\sigma} R^{\mu\nu\xi\sigma} + d \square R \right),$$

where  $\kappa^2 > 0$ ,  $\Lambda, a, b, c$  and  $d$  are running coupling constants depending on the Ricci scalar.

# INFLATIONARY UNIVERSE FROM HIGHER DERIVATIVE QUANTUM GRAVITY COUPLED WITH SCALAR ELECTRODYNAMICS

We deal with the renormalization-group (RG) improved effective potential for a massless scalar electrodynamics in curved space-time. The action reads,

$$I = \int_{\mathcal{M}} \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \Lambda + a_1 R^2 + a_2 C_{\mu\nu\xi\sigma} C^{\mu\nu\xi\sigma} + \mathcal{L}_m \right].$$

At the early-time universe, the matter Lagrangian contains gauge fields, scalar multiplets and spinors and the related interactions typical of any Grand Unified Theory (GUT). We consider massless scalar quantum electrodynamics (QED),

$$\mathcal{L}_m = -D_\mu \phi D^\mu \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \xi R \phi^2 - \frac{1}{4!} f \phi^4.$$

The effective Lagrangian reads

$$\mathcal{L}_m = -D_\mu \phi D^\mu \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \xi R \phi^2 - \frac{1}{4!} f \phi^4 = -\frac{\partial_\mu \phi \partial^\mu \phi}{2} - V_{\text{eff}}(\phi, R),$$

where  $\phi = \sqrt{|\phi|}$ , while the effective potential  $V_{\text{eff}} \equiv V_{\text{eff}}(\phi, R)$  has to be evaluated in one-loop approximation in the background where  $\phi$  and  $R$  are almost constants. It satisfies the standard RG equation,

$$\left[ \mu \frac{\partial}{\partial \mu} + \beta_{e^2}(t') \frac{\partial}{\partial e^2(t')} + \beta_f(t') \frac{\partial}{\partial f(t')} + \beta_\xi(t') \frac{\partial}{\partial \xi(t')} - \gamma(t') \phi(t') \frac{\partial}{\partial \phi(t')} \right] V_{\text{eff}} = 0.$$

In this expression, couplings  $e^2(t')$ ,  $f(t')$ ,  $\xi(t')$  and  $\phi(t')$  are the functions of the renormalization parameter  $t'$  given by

$$t' = \frac{1}{2} \log \left[ \frac{\phi^2}{\mu^2} \right],$$

where  $\mu$  is a mass parameter in the range  $\mu \sim \mu_{GUT} = 10^{15} \text{ GeV}$ .

After the RG procedure one obtains:

$$V_{\text{eff}} = -\tilde{f}\phi^4 - A\phi^4 \left[ \log \frac{\phi^2}{\mu^2} - \frac{25}{6} \right] + \tilde{\xi}R\phi^2 - BR\phi^2 \left[ \log \frac{\phi^2}{\mu^2} - 3 \right],$$

with

$$\tilde{f} = \frac{f}{4!}, \quad \tilde{\xi} = \frac{\xi}{2}, \quad A = \frac{1}{48(4\pi)^2} \left( \frac{10}{3}f^2 + 36e^4 \right),$$

$$B = \frac{1}{12(4\pi)^2} \left[ \left( \xi - \frac{1}{6} \right) \left( \frac{4f}{3} - 6e^2 \right) + 6\xi e^2 \right].$$

Now we can study inflation from scalar quantum electrodynamics in the gravitational background presented before with  $\Lambda, C^2 = 0$  (the square of the Weyl tensor does not give any contribution on FLRW space-time), namely in GR +  $a_1 R^2$ -corrections. On FLRW space-time one gets:

$$H_{\text{dS}}^2 \simeq \frac{\left[ \tilde{f} + A \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - \frac{25}{6} \right] \right] \kappa^2 \phi^4}{-3 + 6 \left[ \tilde{\xi} - B \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - 3 \right] \right] \kappa^2 \phi^2}.$$

The field moves as

$$\dot{\phi} \simeq \frac{2\phi \left[ 12H^2 \left[ -2B - \tilde{\xi} + B \log \left[ \frac{\phi^2}{\mu^2} \right] \right] + \left[ -22A/3 + 2\tilde{f} + 2A \log \left[ \frac{\phi^2}{\mu^2} \right] \right] \phi^2 \right]}{3H}.$$

One can introduce the e-folds number  $N := \log[a(t_f)/a(t_i)]$ ,

$$N = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi \simeq \frac{1}{8} \kappa^2 \phi_i^2.$$

To study perturbations left at the end of inflation, one needs the “slow-roll” parameters,


$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_3 = \frac{\dot{F}}{2HF}, \quad \epsilon_4 = \frac{\dot{E}}{2HE},$$

where

$$E = F + \frac{3\dot{F}^2}{2\dot{\phi}^2}, \quad F \equiv F(R, \phi) = \frac{1}{\kappa^2} + 4a_1 R - 2 \frac{dV_{\text{eff}}}{dR}.$$

The spectral index  $n_s$  and the tensor-to-scalar ratio  $r$  read,

$$n_s = 1 - 4\epsilon_1 - 2\epsilon_2 + 2\epsilon_3 - 2\epsilon_4, \quad r = 16(\epsilon_1 + \epsilon_3),$$

where  $\epsilon_{1,2,3,4}$  must be evaluated in the limit  $N \simeq 55 - 65$ . 

We obtain:

$$\begin{aligned}(1 - n_s) &\simeq \frac{2(1 + B/\tilde{\xi})}{N} + \frac{A(192a_1\tilde{f} + (5 - 48\tilde{\xi})\tilde{\xi})}{2\tilde{f}(48a_1\tilde{f} + \tilde{\xi} - 12\tilde{\xi}^2)N}, \\ r &\simeq \frac{\tilde{\xi}}{(4a_1\tilde{f} - \tilde{\xi}^2)N^2} - \frac{16A\tilde{\xi}^2}{\tilde{f}(4a_1\tilde{f} - \tilde{\xi}^2)N} - \frac{32B\tilde{\xi}}{(4a_1\tilde{f} - \tilde{\xi}^2)N} \quad (1)\end{aligned}$$

The recent Planck satellite results constraint these quantities as  $n_s = 0.968 \pm 0.006$  (68% CL) and  $r < 0.11$  (95% CL). Moreover, the last BICEP2/Keck Array data yield a (combined) upper limit for the tensor-to-scalar ratio as  $r < 0.07$  (95% CL). If one takes  $N \sim 55 - 65$ , in the limit  $A = B = 0$ , the tensor-to-scalar ratio is small enough to satisfy the Planck and the BICEP2/Keck Array data, while the spectral index is in agreement with the Planck results inside the given range. Thus, the one-loop potential slightly changes these indexes, and the model is viable as long as  $|B/\tilde{\xi}|, |A/\tilde{f}| \ll 1$ .

# INFLATION IN QG WITH SCALAR QUANTUM ELECTRODYNAMICS

The general formalism of a RG-improved theory requires an explicit dependence on the renormalization scale of  $\kappa^2 \equiv \kappa^2(t')$ ,  $a_1 \equiv a_1(t')$  and  $\Lambda \equiv \Lambda(t')$  inside the action. We neglect the contribute of the Cosmological constant and

$$\frac{1}{\kappa^2(t')} = \frac{1}{\tilde{\kappa}^2} - 2m^2 t', \quad a_1(t') \equiv \tilde{a}_1 + 2b_1 t',$$

where  $\tilde{a}_1 = a_1(t' = 0)$ ,  $b_1$  is an adimensional parameter and  $m^2$  a mass constant such that (during inflation),

$$m^2 < \frac{1}{2\tilde{\kappa}^2 t'}.$$

The (quasi) de Sitter solution is given by

$$H_{\text{dS}}^2 \simeq \frac{\left[ \tilde{f} + A \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - \frac{25}{6} \right] \right] \tilde{\kappa}^2 \phi^4}{-3 \left[ 1 - m^2 \tilde{\kappa}^2 \log \left[ \frac{\phi^2}{\mu^2} \right] \right] + 6 \left[ \tilde{\xi} - B \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - 3 \right] \right] \tilde{\kappa}^2 \phi^2}.$$



The spectral index and the tensor-to-scalar ratio of the model are derived as:

$$(1 - n_s) \simeq \frac{2 \left[ 8 - 8m^2 \tilde{\kappa}^2 \left[ -2 + \log \left[ \frac{\phi^2}{\mu^2} \right] \right] \right]}{\tilde{\kappa}^2 \phi^2},$$

$$r \simeq \frac{64 \tilde{\xi} \left[ 1 - m^2 \tilde{\kappa}^2 \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - 1 \right] \right]^2}{(4a_1 \tilde{f} - \tilde{\xi}^2) \tilde{\kappa}^4 \phi^4},$$

with

$$N \simeq \frac{\kappa^2 \phi_i^2}{8 - 8m^2 \tilde{\kappa}^2 \left[ \log \left[ \frac{\phi_i^2}{\mu^2} \right] - 1 \right]}.$$

In order to satisfy the last Planck satellite results it must be

$$\frac{2}{(1 - n_s)} = \frac{\tilde{\kappa}^2 \phi^2}{\left[ 8 - 8m^2 \tilde{\kappa}^2 \left[ \log \left[ \frac{\phi^2}{\mu^2} \right] - 2 \right] \right]} \simeq 60.$$

In this case, the spectral index lies inside the observed range, while the tensor-to-scalar ratio is small enough to be in agreement with Planck and BICEP2/Keck Array data.

# BEYOND-ONE-LOOP QUANTUM GRAVITY ACTION YIELDING BOTH INFLATION AND LATE-TIME ACCELERATION

The action reads

$$I = \int_{\mathcal{M}} d^4 \sqrt{-g} \left[ \frac{R}{\kappa^2(t')} - \tilde{\Lambda}(t') - \frac{\omega(t')}{3\lambda(t')} R^2 + \frac{1}{\lambda(t')} C^2 + f_{\text{DE}}(R) + \mathcal{L}_m \right],$$

where  $f_{\text{DE}}(R)$  is phenomenological  $F(R)$ -gravity introduced to reproduce the late-time acceleration. The inflationary scenario is still realized by high-curvature quantum corrections to gravity. The one-loop coupling constants obey to RG equations and are expressed in terms of the log term of a characteristic mass scale in the theory, namely

$$t' = \frac{t'_0}{2} \log \left[ \frac{R}{R_0} \right]^2, \quad 0 < t'_0,$$

where  $R_0$  is the curvature at the end of inflation.

For the running coupling constant we find:

$$\lambda(t') = \frac{\lambda(0)}{1 + \lambda(0)\beta_2 t'}, \quad \omega(t') \simeq \omega_1, \quad \kappa^2(t') = \kappa_0^2(1 + \lambda(0)\beta_2 t')^{Z/\beta_2},$$

$$\kappa^4 \tilde{\Lambda} = -\frac{3\lambda(0)(1 + 20\omega_1^2)}{4\omega_1(1 + \lambda(0)\beta_2 t')(-1 + 30\omega_1 + 6\beta_2\omega_1 + 40\omega_1^2)} + \kappa_0^4 \tilde{\Lambda}_0(1 + \lambda(0)\beta_2 t')^W$$

with  $\kappa_0 = 8\pi/M_{Pl}^2$ ,  $\beta_2 = 133/10$ ,  $\omega_1 = -0.02$ ,  $Z = 10.27$ ,  $W = 13.2$ . In order to avoid the effects of  $\tilde{\Lambda}$  at low curvature, we fix  $\tilde{\Lambda}_0$  such that  $\tilde{\Lambda}(t' = 0) = 0$ . Moreover, it must be

$$\lambda(0) \ll \left| \frac{1}{\beta_2 t'_0 \log[4\Lambda/R_0]} \right|,$$

and the QG effects are negligible when  $R \ll R_0$ . We also observe that the Weyl term does not give contribution on FRW space-time.

## INFLATION

The de Sitter solution of inflation reads,

$$H_{\text{dS}}^2 k_0^2 \simeq \frac{0.107}{t_0' (\lambda(0) t')^{0.77}},$$

while the Hubble parameter evolves as

$$H \simeq H_{\text{dS}} \left( 1 - e^{-\frac{A_+ N}{H_{\text{dS}}}} \right), \quad A_+ \simeq 1.486 \left( \frac{H_{\text{dS}} t_0'}{t'} \right),$$

or, in terms of the  $e$ -folds,

$$H \simeq H_{\text{dS}} \left( 1 - e^{-\frac{N}{\mathcal{N}_{\text{min}}}} \right), \quad \mathcal{N}_{\text{min}} \simeq (1.486)^{-1} \log \left[ \frac{R_{\text{dS}}}{R_0} \right].$$

The spectral index and the tensor-to-scalar ratio lead to

$$(1 - n_s) = \frac{2}{\mathcal{N}_{\text{min}}}, \quad r = \frac{48}{\mathcal{N}_{\text{min}}^2} e^{-\frac{2N}{\mathcal{N}_{\text{min}}}},$$

and the model is viable for  $\mathcal{N}_{\text{min}} \simeq 60$ ,  $\mathcal{N}_{\text{min}} \leq N$ .

## DARK ENERGY

For the dark energy sector we propose the following form of exponential gravity,

$$f_{\text{DE}}(R) = -\frac{2\Lambda g(R)(1 - e^{-bR/\Lambda})}{\kappa^2}, \quad 0 < b,$$

where  $g(R)$  is necessary to stabilize the theory at large red-shift and explicitly reads,

$$g(R) = \left[ 1 - c \left( \frac{R}{4\Lambda} \right) \log \left[ \frac{R}{4\Lambda} \right] \right], \quad 0 < c,$$

where  $c$  is a positive parameter. Exponential gravity mimics the Cosmological Constant at high curvatures and pass all the cosmological tests. Moreover, for  $R = 0$ , we recover the Minkowski space-time.

During matter/radiation era, the dark energy density oscillates with frequency

$$\nu \simeq \frac{1}{2\pi\sqrt{Rf_{\text{DE}}(R)}(z+1)}, \quad z = \frac{1}{a(t)} - 1,$$

and the theory suffers of the singularity problem when  $f_{\text{DE}}(R) \simeq 0^+$ . In our case, thanks to the suitable function  $g(R)$ , we find

$$\nu \simeq \frac{\sqrt{2/b}}{2\pi(z+1)},$$

and the oscillation frequency of dark energy decreases at high red-shift.

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