Fun with thermal dimension-six operators[1]

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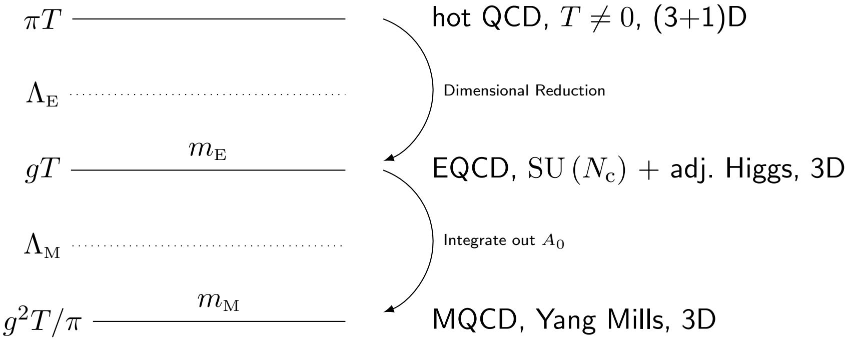
[1] M. Laine, P. Schicho, and Y. Schröder. Soft thermal contributions to 3-loop gauge coupling. *J. High Energy Phys.* **2018**, 388 (2018) .

EFT construction

- "hard scale" $\sim \pi T$
- "soft scale" $\sim m_{\mathrm{E}} \sim gT$
- "ultrasoft scale" $\sim g^2 T/\pi$



Dimensionally reduced effective theory for hot QCD

Super renormalisable truncation "Electrostatic QCD" (EQCD)

$$L_{\mathrm{EQCD}} \equiv \frac{1}{4}F^a_{ij}F^a_{ij} + \frac{1}{2}\mathcal{D}^{ab}_i A^b_0 \mathcal{D}^{ac}_i A^c_0 + \frac{1}{2}m^2_{\mathrm{E}} A^a_0 A^a_0 + \frac{1}{4!}\lambda_{\mathrm{E}} X^{abcd} A^a_0 A^b_0 A^c_0 A^d_0 \ .$$

Developed for studying high-temperature thermodynamics.[2] Successful with soft light-cone observables.[3]

Color trace in adjoint rep.[4]

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[2] P. H. Ginsparg. First Order and Second Order Phase Transitions in Gauge Theories at Finite Temperature. *Nucl. Phys.* **B170**, 388–408 (1980) , T. Appelquist and R. D. Pisarski. High-temperature Yang-Mills theories and three-dimensional quantum chromodynamics. *Physical Review D* **23**, 2305–2317 (1981) .

[3] See talk by J. Ghiglieri [Wed 11:45]

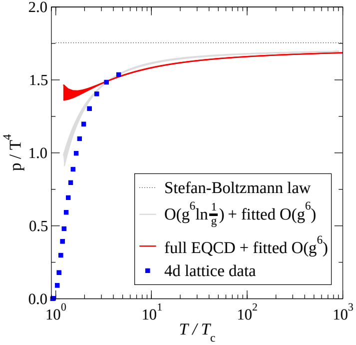
[4] $\mathrm{Tr}\,(AB) = A_{ab}B_{ba}$ with e.g. $(A_0)_{ab} = -if^{abc}A^c_0$ and $X^{abcd} = f^{m_4 a m_1} \cdots f^{m_3 d m_4}$ etc.

Motivation

EQCD fails to describe hot QCD close to T_c

Pure gauge phase transition governed by Z(3) center symmetry breaking. Non-perturbative EQCD dynamics lacking satisfactory results on hot QCD pressure.[5]

Non-perturbative computation in EQCD decreases fit quality further. What kind of effects are responsible for this mismatch?



[5] A. Hietanen et al. Three-dimensional physics and the pressure of hot QCD. *Phys. Rev. D* **79**, 045018 (2009) .

Possible improvements:

- More precise computation of parameters.
- Inclusion of higher-dimensional operators.

These may even be related to each other.

3-loop hard correction to $g_{\rm E}^2$:[6]

Renormalised NNLO result for effective gauge coupling

$$g_{\rm E}^2 = g^2(\mathcal{Z}_B + \delta\mathcal{Z}_B)^{-1} \ .$$

$$\Gamma^{(2)}_{\rm EQCD}[B] = \frac{1}{2}\, B_i^a(q)\, B_j^b(r)\, \delta^{ab}\, \delta(q+r)\, \left(q^2\delta_{ij} - q_i q_j\right)\, (\mathcal{Z}_B + \delta\mathcal{Z}_B) \ ,$$

$$\mathcal{Z}_B = 1 - \frac{g^2 N_{\rm c}}{(4\pi)^2}\left[\frac{22}{3}L + \frac{1}{3}\right] - \frac{g^4 N_{\rm c}^2}{(4\pi)^4}\left[\frac{68}{3}L + \frac{341}{18} - \frac{10\zeta_3}{9}\right] - \frac{g^6 N_{\rm c}^3}{(4\pi)^6}\left[\frac{748}{9}L^2 + \left(\frac{6608}{27} - \frac{10982\zeta_3}{135}\right)L + \text{(finite)}\right] + \mathcal{O}(g^8) \ ,$$

$$\delta\mathcal{Z}_B = \frac{g^6 N_{\rm c}^3}{(4\pi)^6}\,\frac{61\zeta_3}{5\epsilon} + \mathcal{O}(g^8) \ , \quad L = \ln\left(\frac{\bar{\mu}e^{\gamma_{\rm E}}}{4\pi T}\right) \ .$$

How to interpret leftover IR divergence even after renormalisation?

[6] I. Ghisoiu. Three-loop Debye mass and effective coupling in thermal QCD. (2013) , I. Ghisoiu and Y. Schröder. Three-loop Debye mass and effective coupling in thermal QCD. In *SEWM14* (2014).

Dimension-six operators

Dimension-six operators in EQCD: Chapman Vertices[7]

1-loop sum-integral yields missing ζ_3 contributions and is finite

$$\sum\!\!\!\!\!\!\!\int_P' \frac{1}{P^6} = \frac{\zeta_3}{128\pi^4 T^2}\left[1+\mathcal{O}(\epsilon)\right] .$$

Augment L_{EQCD} with $d=6$ operators:

$$\begin{aligned}\delta L_{\mathrm{EQCD}}[A] = \left(\frac{2g_{\mathrm{E}}^2\zeta_3}{128\pi^4T^2}\right) \mathrm{Tr}\, \Big\{ & c_1\,(D_\mu F_{\mu\nu})^2 + c_2\,(D_\mu F_{\mu 0})^2 \\ & + i g_{\mathrm{E}} \left[c_3\, F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} + c_4\, F_{0\mu}F_{\mu\nu}F_{\nu 0} + c_5\, A_0 (D_\mu F_{\mu\nu}) F_{0\nu}\right] \\ & + i g_{\mathrm{E}}^2 \Big[c_6\, A_0^2 F_{\mu\nu}^2 + c_7\, A_0 F_{\mu\nu} A_0 F_{\mu\nu} \\ & \quad + c_8\, A_0^2 F_{0\mu}^2 + c_9\, A_0 F_{0\mu} A_0 F_{0\mu}\Big] \\ & + g_{\mathrm{E}}^4 \left[c_{10} A_0^6\right] \Big\} .\end{aligned}$$

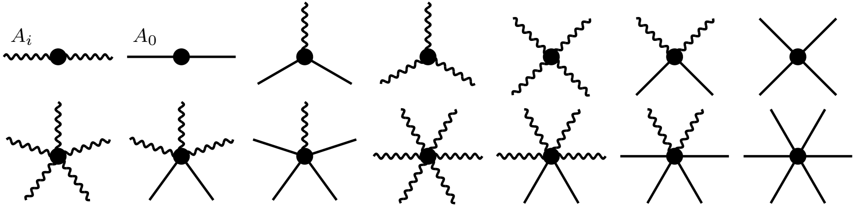
Use redundancy for cross-checks with no physical effect on

$$c_i^{\mathrm{new}} \equiv c_i + \delta c_i, \quad i = 4,\ldots,7 .$$

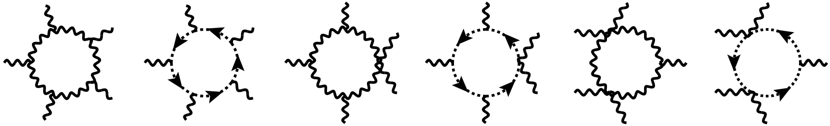
[7] S. Chapman. New dimensionally reduced effective action for QCD at high temperature. *Phys. Rev. D* **50**, 5308–5313 (1994) .

Vertex structures

L_{EQCD} becomes non-renormalizable



Use background field gauge (bfg)[8] to determine coefficients c_i in d dimensions by evaluating 2-,3-,5- and 6-point vertices at one-loop order in hot YM $\Rightarrow$ uniqueness.



[8] L. F. Abbott. The background field method beyond one loop. *Nuclear Physics B* **185**, 189–203 (1981) .

Coefficients

LO coefficients of dim-6 operators ($\alpha =$ gauge parameter)

$$
\begin{aligned}
c_1 &= \frac{41-d}{120} + \frac{(8-\alpha)\alpha}{48} \,, \\
c_2 &= \frac{(d-1)(d-5)}{120} + \frac{(d-5)(4+\alpha)\alpha}{48} \,, \quad c_3 = \frac{1-d}{180} \,, \\
c_4 - 2c_7 &= \frac{(41-d)(5-d)}{60} \,, \quad c_6 + c_7 = \frac{(d-25)(5-d)}{24} \,, \\
c_5 - 2c_7 &= \frac{(21-d)(5-d)}{30} + \frac{(d-5)\alpha}{6} \,, \\
c_8 &= \frac{(5-d)(3-d)(d-1)}{20} + \frac{(d-5)(d-3)\alpha}{3} \,, \\
c_9 &= \frac{(5-d)(3-d)(d-1)}{30} + \frac{(d-5)(d-3)\alpha}{6} \,, \\
c_{10} &= \frac{(d-5)(d-3)(d-1)^2}{180} \,.
\end{aligned}
$$

In $d = 3 - 2\epsilon \Rightarrow c_8, c_9, c_{10}$ couple to evanescent operators.

Loop contributions

Integrate out $m_{\rm E}$: soft/hard effects

Specifically, compute gauge coupling of "Magnetostatic Modes"

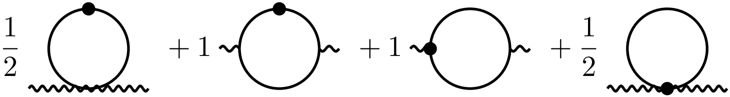
$$L_{\rm MQCD} \equiv \frac{1}{4} F^a_{ij} F^a_{ij} \ .$$

The MQCD coupling $g^2_{\rm M}$ is defined via 2-point function

$$\Gamma^{(2)}_{\rm MQCD}[B] = \frac{1}{2} B^a_\mu(q)\, B^b_\nu(-q) \left(q^2 \delta_{ij} - q_i q_j\right) (Z_B + \delta Z_B) \ ,$$

$$g^2_{\rm M} = g^2_{\rm E} (Z_B + \delta Z_B)^{-1} \ .$$

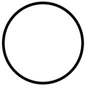
1-loop level



1-loop diagrams with blobs as Chapman vertex insertion

$$\delta\Gamma^{(2)}_{\rm MQCD}[B] = B^a_i(q)\, B^b_j(r)\, \delta^{ab}\, \delta(q+r)\, (q^2\delta_{ij} - q_i q_j) \left(\frac{g_{\rm E}^4 N_{\rm c}^2 \zeta_3}{128\pi^4 T^2}\right) I(m_{\rm E})$$
$$\times \left\{ \frac{(4-d)(d-2)}{12}\,(c_1+c_2) + 3c_3 + (c_4 - 2c_7) + 4(c_6+c_7) \right\} .$$

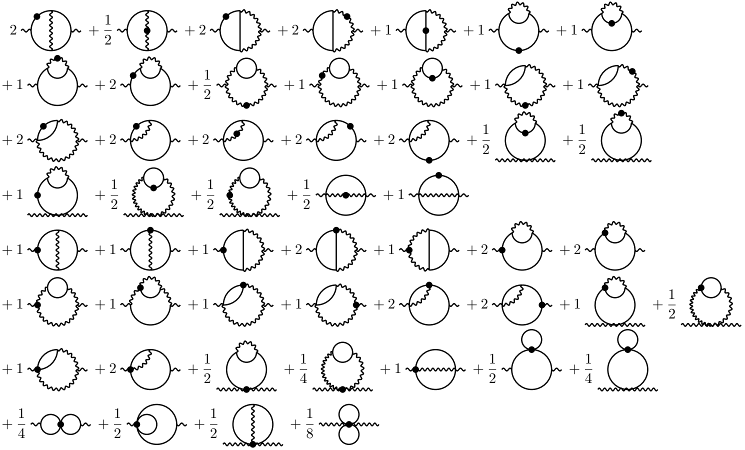
Simplest vacuum diagram tadpole

$I(m_{\rm E}) =$  $\equiv \int_p \frac{T}{p^2+m_{\rm E}^2} = -\frac{\mu^{-2\epsilon} m_{\rm E} T}{4\pi}\,[1+\mathcal{O}(\epsilon)]$

yields finite term $\mathcal{O}(g^4 m_{\rm E}/T) \sim \mathcal{O}(g^5) \to$ Largest soft contribution.

2-loop level

From 2pt function in bfg; need at least 2-loop for div (3d).



Divergence from 1+2-loop

$$
\begin{aligned}
Z_B &= 1 + \left(\frac{g_{\rm E}^2 N_{\rm c}}{16\pi^2}\right)^2 \frac{m_{\rm E}}{2\pi T}\left(\frac{875\zeta_3}{72}\right) \\
&\quad - \left(\frac{g_{\rm E}^2 N_{\rm c}}{16\pi^2}\right)^3 \left(\frac{1097\zeta_3}{549}\right)\frac{61}{5}\left\{L + 2\ln\left(\frac{\bar\mu}{2m_{\rm E}}\right) + \frac{\zeta_3'}{\zeta_3} - \gamma_{\rm E} + \frac{103771}{52656}\right\}\,, \\
\delta Z_B &= \left(\frac{g_{\rm E}^2 N_{\rm c}}{16\pi^2}\right)^3 \left(-\frac{1097}{1098}\right)\frac{61\zeta_3}{5\epsilon}\,.
\end{aligned}
$$

Almost perfect cancellation of $\frac{1097}{1098}$ of the hard-mode divergence after integrating out hard $\sim \pi T$ and soft scale $\sim gT$.

Intermediate summary

Set $g_{\rm ER} = g^2(1+\mathcal{O}(g^2))$, parameterise remaining $\frac{1}{1098}$ of IR divergence and combine with results from hard modes

$$\delta\mathcal{Z}_B + \delta Z_B \;=\; \frac{g^6 N_{\rm c}^3 T^2}{(8\pi)^2}\left(\frac{\zeta_3}{128\pi^4 T^2}\right)\frac{1}{45\epsilon} + \mathcal{O}(g^8)\;.$$

Anything missing? Ultrasoft Contributions!

Remains to check contributions from ultrasoft $\sim g^2T/\pi$ scales with dim-6 operators in MQCD:

$$\delta L_{\rm MQCD}[A] = \left(\frac{2g_{\rm M}^2\zeta_3}{128\pi^4T^2}\right) {\rm Tr}\, \left\{c_1\, (D_iF_{ij})^2 + ig_{\rm M}c_3\, F_{ij}F_{jk}F_{ki}\right\} \, .$$

Evaluate UV divergence using IR cut-off by introducing fictitious mass $m_{\rm G}$ (unphysical, only need div):

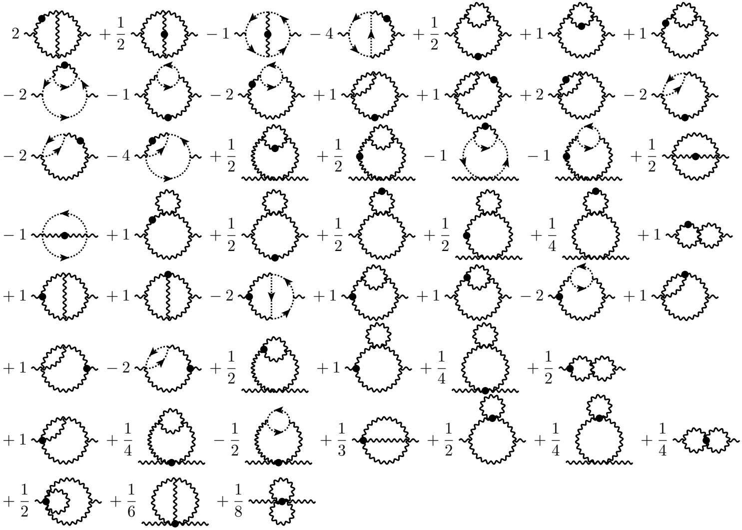
$$\left\langle A_k^a(p)A_l^b(q)\right\rangle \equiv \frac{\delta^{ab}\delta(p+q)}{p^2+m_{\rm G}^2}\left(\delta_{kl} - (1-\xi)\frac{p_kp_l}{p^2+m_{\rm G}^2}\right) \, ,$$

$$\left\langle c^a(p)\bar{c}^b(q)\right\rangle \equiv \frac{\delta^{ab}\delta(p-q)}{p^2+m_{\rm G}^2} \, .$$

Inspect divergent part and let $m_{\rm G} \to 0$.

2-loop level

MQCD dim-6 contributions to 2pt function in bfg.



Remaining Divergence

Include 2-loop results from dim-6 MQCD:

$$
\begin{aligned}
\delta\Gamma^{(2)}_{\mathrm{IR}}[B] &= \frac{1}{2}\, B^a_i(q)\, B^a_j(-q)\, \left(q^2\delta_{ij} - q_i q_j\right) \\
&\quad \times \left(\frac{g^6_{\mathrm{M}} N^3_{\mathrm{c}} \zeta_3}{128\pi^4 T^2}\right) \frac{T^2 c_3}{(4\pi)^2 2\epsilon}\;, \\
&= \frac{1}{2}\, B^a_i(q)\, B^a_j(-q)\, \left(q^2\delta_{ij} - q_i q_j\right) \\
&\quad \times \frac{g^6 N^3_{\mathrm{c}} T^2}{(8\pi)^2} \left(\frac{\zeta_3}{128\pi^4 T^2}\right) \left(-\frac{1}{45\epsilon}\right) + \mathcal{O}(g^7)\;.
\end{aligned}
$$

- Account for all leftover divergences.
- Result is completely ξ independent.

Conclusion: By combining 2-loop results from from EQCD and MQCD with dim-6 vertices the hard mode result is rendered finite $\Rightarrow$ divergence cancels completely.

Different computation: Purely soft effects

Integrate out m_{E}: soft and ultrasoft/soft effects

Two divergences to obtain $1/m_{\mathrm{E}}^3$-suppression.

3-loop soft computation of 2pt function assumes

$$\widetilde{Z}_B \;=\; 1 + \frac{g_{\mathrm{E}}^2 N_{\mathrm{c}} T}{48\pi m_{\mathrm{E}}} + \left(\frac{g_{\mathrm{E}}^2 N_{\mathrm{c}} T}{16\pi m_{\mathrm{E}}}\right)^2 \left(\frac{19}{18} + \frac{4\lambda}{3}\right) + \mathcal{O}\left(\frac{g_{\mathrm{E}}^2 N_{\mathrm{c}} T}{16\pi m_{\mathrm{E}}}\right)^3 ,$$

while MQCD dim-6 operators take the form

$$\delta L_{\mathrm{MQCD}} = \left(\frac{g_{\mathrm{M}}^2 T}{16\pi m_{\mathrm{E}}^3}\right) \mathrm{Tr}\,\left\{\tilde{c}_1 \,(D_i F_{ij})^2 + i g_{\mathrm{M}} \tilde{c}_3 \, F_{ij} F_{jk} F_{ki}\right\} \; .$$

- There are again log-divergences.
- No cancellation (even after mass renormalisation).
- 3-loop affected by small mass ambiguity at 1-loop level.
- $\mathcal{O}\left(\frac{g_{\mathrm{ER}}^6 N_{\mathrm{c}}^3 T^3}{m_{\mathrm{ER}}^3}\right)$ contribution to g_{M}^2 non-perturbative?

Conclusions

- Soft scale $m_{\mathrm{E}} \sim gT$ formally larger than ultrasoft scale $\sim g^2T/\pi$.
- Soft scale nevertheless important for IR dynamics.
- It is 1097 times more important than magnetic scale in terms of an IR divergence in the 3-loop gauge coupling.
- Thus, need to include dim-6 Chapman operators in EQCD for good precision and accounting for IR physics.

- MQCD non-perturbative at $\mathcal{O}\left(\frac{g_{\mathrm{ER}}^6 T^3}{m_{\mathrm{ER}}^3}\right) \sim \mathcal{O}(g^3)$.

Backup

Overlapping soft/hard and ultrasoft/hard contributions

Power counting: Only keep contributions of form "2-loop soft $\times$ 1-loop hard" (one dim-6 insertion)

- 1-loop hard $\sim g^2/T^2$
- 1-loop soft $\sim g^2 T m_{\rm ER} \sim g^3 T^2$
- 2-loop soft $\sim (g^2 T)^2 \sim g^4 T^2$
- 2-loop hard $\sim g^4/T^2$

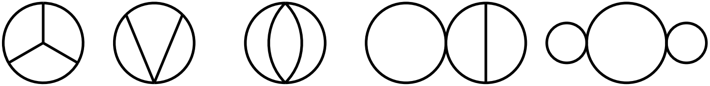
Thus, 2-loop hard $\times$ 1-loop soft" is $\sim g^7$ exceeding resolution of computation $\sim g^6$.

Similar arguments for dimension-8 operators suppressed via soft effects $\sim g^2 T m_{\rm ER}^3 \sim g^5 T^4$ where blobs are $\sim g^2/T^4$.

3-loop EQCD computation

In-house algebraic manipulations with FORM[9] for IBP reduction and diagram generation with QGRAF[10].

3-loop Master diagrams are well known with appearing non-trivial vacuum topologies:



[9] B. Ruijl, T. Ueda, and J. Vermaseren. FORM version 4.2. *arXiv* (2017), [1707.06453].

[10] P. Nogueira. Automatic Feynman Graph Generation. *Journal of Computational Physics* **105**, 279–289 (1993) .

So far...

Integrate out scale $m_{\rm E}$ directly (no Chapman vertices).

Notation: Distinguish contributions as $\widetilde{Z}_B + \delta\widetilde{Z}_B$.

2-loop result[11]

$$\widetilde{Z}_B \;=\; 1 + \left(\frac{g_{\rm E}^2 N_{\rm c} T}{16\pi m_{\rm E}}\right)\frac{1}{3} + \left(\frac{g_{\rm E}^2 N_{\rm c} T}{16\pi m_{\rm E}}\right)^2\left(\frac{19}{18} + \frac{4\lambda}{3}\right) \;.$$

[11] P. Giovannangeli. Two loop renormalization of the magnetic coupling and non-perturbative sector in hot QCD. *Nucl. Phys. B* **738**, 23–47. 32 p (2005) .

Pure 3-loop EQCD result

$$\widetilde{Z}_B^{(3)} + \delta\widetilde{Z}_B^{(3)} = \left(\frac{g_{\mathrm{E}}^2 N_{\mathrm{c}} T}{16\pi m_{\mathrm{E}}}\right)^3 \left[\frac{1}{6\epsilon} + \ln\left(\frac{\bar{\mu}}{2m_{\mathrm{E}}}\right) + \frac{2}{945}\left(23510 + 12600\zeta_2 - 1101\ln(2)\right) + \frac{2\kappa_2 - 8\lambda}{3\epsilon} + 6(2\kappa_2 - 8\lambda)\ln\left(\frac{\bar{\mu}}{2m_{\mathrm{E}}}\right) + \frac{1}{9}\left(4\lambda + 24\lambda^2 - \kappa_1(5 - 8\ln 2) + \kappa_2(31 - 24\ln 2)\right)\right] .$$

Note the leftover divergence after (mass) renormalisation[12]

$$\delta g_{\mathrm{E}}^2 = 0\,, \quad \delta m_{\mathrm{E}}^2 = 2\left(\frac{g_{\mathrm{ER}}^2 N_{\mathrm{c}} T}{16\pi}\right)^2 \frac{2\kappa_2 - 8\lambda}{\epsilon}\,,$$

with

$$\lambda = \frac{5N_{\mathrm{c}}\lambda_{\mathrm{E}}}{24 g_{\mathrm{E}}^2}\,, \quad \kappa_1 = \lambda_{\mathrm{E}}\frac{\left(N_{\mathrm{c}}^2 + 36\right)}{12 g_{\mathrm{E}}^2 N_{\mathrm{c}}}\,, \quad \kappa_2 = \lambda_{\mathrm{E}}^2\frac{\left(N_{\mathrm{c}}^2 + 36\right)}{(12 g_{\mathrm{E}}^2)^2}\,.$$

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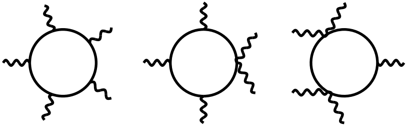
[12] I. Ghisoiu, J. Moller, and Y. Schröder. Debye screening mass of hot Yang-Mills theory to three-loop order. *arXiv*, 141 (2015), [1509.08727].

MQCD augmentation

Include dim-6 operators associated with scale m_{E}

$$\delta L_{\mathrm{MQCD}} = \left(\frac{g_{\mathrm{M}}^2 T}{16\pi m_{\mathrm{E}}^3}\right) \mathrm{Tr}\ \left\{\tilde{c}_1\,(D_i F_{ij})^2 + i g_{\mathrm{M}} \tilde{c}_3\, F_{ij} F_{jk} F_{ki}\right\}\ ,$$

with contributing 5pt diagrams at LO in EQCD:



The coefficients read

$$\tilde{c}_1 = -\frac{1}{120}\ , \quad \tilde{c}_3 = -\frac{1}{180}\ .$$

2-loop contribution from dimension-six MQCD

Same game as before $c_i \to \tilde{c}_i$

$$\delta\widetilde{\Gamma}^{(2)}_{\rm IR}[B] = \frac{1}{2}\, B_i^a(q)\, B_j^a(-q)\, \left(q^2\delta_{ij} - q_i q_j\right) \times \left(\frac{g_{\rm E}^2 N_{\rm c} T}{16\pi m_{\rm E}}\right)^3 \left\{-\frac{1}{45\epsilon} + \text{(finite)}\right\} \,.$$

- Completely independent of gauge parameter ξ.
- Leftover divergence persists.

Immediate implications?

Physical Debye mass is non-perturbative at NLO. However, $m_{\rm E}^2$ is a Langrangian parameter but might still be IR sensitive at $\mathcal{O}(g^4T^2/\pi^2)$.

- Π_{00} with IR shielding yields UV $\delta m_{\rm E}$.
- Π_{00} without IR shielding gives IR divergence at $\mathcal{O}(g^4T^2/\pi^2)$.

Insertion of ambiguity into 1-loop results and re-expansion up to 3-loop order

$$\frac{g_{\rm ER}^2 N_{\rm c} T}{48\pi\left[m_{\rm ER}^2 + \frac{\beta}{\epsilon_{\rm IR}}\left(\frac{g_{\rm ER}^2 N_{\rm c} T}{16\pi}\right)^2\right]^{1/2}} \leftrightarrow -\frac{\beta}{6\epsilon_{\rm IR}}\left(\frac{g_{\rm ER}^2 N_{\rm c} T}{16\pi m_{\rm ER}}\right)^3$$

and $\beta = -\frac{13}{15}$.

Mass ambiguity prohibits purely perturbative determination.