Hydro+: Hydrodynamics for QCD critical point

M. Stephanov

with Y. Yin (MIT), 1712.10305
Critical point
– end of phase coexistence – is a ubiquitous phenomenon

Water:

<table>
<thead>
<tr>
<th>Substance</th>
<th>Critical temperature °C</th>
<th>Critical pressure (absolute) atm</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argon</td>
<td>-122.4 °C (150.8 K)</td>
<td>48.1</td>
<td></td>
</tr>
<tr>
<td>Ammonia[^1][^15]</td>
<td>132.4 °C (405.5 K)</td>
<td>111.3 (11,280 kPa)</td>
<td></td>
</tr>
<tr>
<td>Bromine</td>
<td>310.8 °C (584.0 K)</td>
<td>102</td>
<td></td>
</tr>
<tr>
<td>Caesium</td>
<td>1,664.85 °C (1,938.00 K)</td>
<td>94</td>
<td></td>
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<tr>
<td>Chlorine</td>
<td>143.8 °C (416.9 K)</td>
<td>76.0</td>
<td></td>
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<tr>
<td>Ethanol</td>
<td>241 °C (514 K)</td>
<td>62.18</td>
<td></td>
</tr>
<tr>
<td>Fluorine</td>
<td>-128.85 °C (144.30 K)</td>
<td>51.5 (5,220 kPa)</td>
<td></td>
</tr>
<tr>
<td>Helium</td>
<td>-267.96 °C (5.19 K)</td>
<td>2.24</td>
<td></td>
</tr>
<tr>
<td>Hydrogen</td>
<td>-239.95 °C (33.20 K)</td>
<td>12.8</td>
<td></td>
</tr>
<tr>
<td>Krypton</td>
<td>-63.8 °C (209.3 K)</td>
<td>54.3</td>
<td></td>
</tr>
<tr>
<td>CH₄ (methane)</td>
<td>-82.3 °C (190.8 K)</td>
<td>45.79</td>
<td></td>
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<tr>
<td>Neon</td>
<td>-228.75 °C (44.40 K)</td>
<td>27.2 (2,760 kPa)</td>
<td></td>
</tr>
<tr>
<td>Nitrogen</td>
<td>-146.9 °C (126.2 K)</td>
<td>33.5</td>
<td></td>
</tr>
<tr>
<td>Oxygen</td>
<td>-118.6 °C (154.2 K)</td>
<td>49.8</td>
<td></td>
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<tr>
<td>CO₂</td>
<td>31.04 °C (304.19 K)</td>
<td>72.8</td>
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<tr>
<td>N₂O</td>
<td>36.4 °C (309.5 K)</td>
<td>71.5</td>
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<tr>
<td>H₂SO₄</td>
<td>654 °C (927 K)</td>
<td>45.4</td>
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<tr>
<td>Xenon</td>
<td>16.6 °C (289.8 K)</td>
<td>57.6</td>
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<tr>
<td>Lithium</td>
<td>2,950 °C (3,320 K)</td>
<td>652</td>
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<tr>
<td>Mercury</td>
<td>1,476.9 °C (1,750.1 K)</td>
<td>1,720 (174,000 kPa)</td>
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<tr>
<td>Sulfur</td>
<td>1,040.85 °C (1,314.00 K)</td>
<td>207</td>
<td></td>
</tr>
<tr>
<td>Iron</td>
<td>8,227 °C (8,500 K)</td>
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</tr>
<tr>
<td>Gold</td>
<td>6,977 °C (7,250 K)</td>
<td>5,000 (510,000 kPa)</td>
<td></td>
</tr>
<tr>
<td>Water[^2][^16]</td>
<td>373.946 °C (647.096 K)</td>
<td>217.7 (22.06 MPa)</td>
<td></td>
</tr>
</tbody>
</table>

Is there one in QCD?
QCD critical point

- QCD is a relativistic QFT of a fundamental force, not quite like non-relativistic fluids.

- But a critical point is a very universal phenomenon – it takes 2 phases whose coexistence (first-order transition) ends.
QCD critical point

QCD is a relativistic QFT of a fundamental force, not quite like non-relativistic fluids.

But a critical point is a very universal phenomenon – it takes 2 phases whose coexistence (first-order transition) ends.

In QCD:

The two phases: quark-gluon plasma and hadron gas.
Experiments: QGP has liquid properties – almost perfect fluidity.

If the phases are separated by a first-order phase transition, there must also be a critical point!
Lattice QCD at $\mu_B \lesssim 2T$ – a crossover

Therefore, if at larger $\mu_B$ ∃ first-order transition ⇒ ∃ critical point.
Lattice QCD at $\mu_B \lesssim 2T$ – a crossover

Therefore, if at larger $\mu_B$ $\exists$ first-order transition $\Rightarrow \exists$ critical point
QCD phase diagram (sketch)

Lattice QCD at $\mu_B \lesssim 2T$ – a crossover

Therefore, if at larger $\mu_B \exists$ first-order transition $\iff \exists$ critical point
Critical point discovery challenges

Essentially two approaches to discovering the QCD critical point.

Each with its own challenges.

- Heavy-ion collisions.

Encouraging progress and intriguing new results.

Challenge in connecting the two: *non-equilibrium dynamics.*
Fluctuations as signatures of the critical point

Fluctuations are observables on the lattice and in heavy-ion collisions.

The key equation:

\[ P(\sigma) \sim e^{S(\sigma)} \quad (Einstein \ 1910) \]
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\[ \delta\sigma \sim \frac{1}{\sqrt{V}} \]

\( \delta\sigma \) is not an average of \( \infty \) many uncorrelated contributions:

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\[ \delta \sigma \sim \frac{1}{\sqrt{V}} \]

\( \chi \sim 1 \quad \chi \sim V^{2/3} \quad \chi \sim 1 \)

CLT?

\( \delta \sigma \) is not an average of \( \infty \) many uncorrelated contributions: \( \xi \rightarrow \infty \)

In fact, \( \langle \delta \sigma^2 \rangle \sim \xi^2 / V \).
Higher order cumulants

- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on $\xi$.
  
  E.g., $\langle \sigma^2 \rangle \sim \xi^2$ while $\kappa_4 = \langle \sigma^4 \rangle_c \sim \xi^7$ \[PRL102(2009)032301\]

- For $n > 2$, sign depends on which side of the CP we are.
  
  This dependence is also universal. \[PRL107(2011)052301\]

- Using Ising model variables:

\[\kappa_4 < 0\]
\[\kappa_4 > 0\]

far from CP:

\[\kappa_4 = 0\]

1st order side

\[\kappa_4 < 0\]

1st order side

\[\kappa_4 > 0\]
Mapping Ising to QCD phase diagram

Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:

In QCD \( (t, H) \rightarrow (\mu - \mu_{CP}, T - T_{CP}) \)

\( \kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma)) \)
Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:
Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:

- $\sqrt{s}$ vs $\omega_4$
- $\mu_B$ vs $\kappa_4$ vs $T$ and $\mu_B$

Graphs showing:
- Million events vs $s_{NN}$
- $dN/dy$ vs $y$
- $\kappa_4$ vs $\mu_B$
- $\sigma^2$-net-proton vs Baryon Doping - $\mu_B$ (MeV)

Intriguing hints (2015 LRPNS)

M. Stephanov
Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:

“intriguing hint” (2015 LRPNS)
Non-equilibrium physics is essential near the critical point.

The challenge taken on by BEST COLLABORATION

Goal: build a *quantitative* theoretical framework describing critical point signatures for comparison with experiment.

Strategy:

- Parameterize QCD equation of state with unknown $T_{CP}$ and $\mu_{CP}$ as variable parameters.
- Use it in a hydrodynamic simulation and compare with experiment to determine or constrain $T_{CP}$ and $\mu_{CP}$.
Parameterized EOS for hydro simulations

Parotto et al, 1805.05249

- Variable parameters ($T_{CP}$, $\mu_{CP}$, slopes, etc.) control Ising-QCD mapping near the QCD critical point: $P = P_{\text{Non-Ising}} + P_{\text{Ising}}$.

- Lattice data at $\mu_B = 0$ is matched:
  
  **Decomposition:** Taylor coefficients from Lattice QCD contain an "Ising" contribution and a "Non-Ising" one:

  \[
  T^4 c_n^{\text{LAT}}(T) = T^4 c_n^{\text{Non-Ising}}(T) + T^4 c_n^{\text{Ising}}(T) \quad (\star)
  \]


This EOS is ready to be used in a hydrodynamic simulation.
Hydrodynamics breaks down near the critical point

Hydrodynamics, as an EFT, relies on separation of scales:

Evolution rate (e.g., expansion time, $\mathcal{O}(10)$ fm) much slower than the local equilibration rate (typically, $\mathcal{O}(0.5 - 1)$ fm).
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- Critical slowing down means relaxation time diverges:
  $\tau_{\text{relaxation}} \sim \xi^z (z \approx 3)$.

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  - In fact, magnitude of $\xi$, and thus fluctuations/cumulants $\kappa_n \sim \xi^p$, is estimated using $\xi \sim \tau_{\text{expansion}}^{1/z}$.

- To be more quantitative we need to describe the breakdown of hydro due to critical slowing down.
This is similar to the breakdown of an effective theory when we consider processes faster than some modes (fields) which we integrated out.
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Breakdown of *locality* is manifested in large gradient corrections to pressure due to $\zeta \sim \xi^3 \rightarrow \infty$.

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\[
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\]

Extending hydro by adding the critically slow modes \( \to \text{Hydro+} \)
What are the additional slow modes?

An *equilibrium* thermodynamic state is completely characterized by average values $\bar{\varepsilon}$, $\bar{n}$, . . .

Fluctuations of $\varepsilon$, $n$ are given by eos: $P \sim \exp(S_{eq}(\varepsilon, n))$. 
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Fluctuations of $\varepsilon, n$ are given by eos: $P \sim \exp(S_{eq}(\varepsilon, n))$.

Hydrodynamics describes *partial-equilibrium states*, i.e., equilibrium is only local, because equilibration time $\sim L^2$.

*Fluctuations* in such states are not necessarily in equilibrium.
Nonequilibrium fluctuations

Measures of fluctuations are *additional* variables needed to characterize the partial-equilibrium state.

2-point (and $n$-point) functions of fluctuating hydro variables: $\langle \delta \varepsilon \delta \varepsilon \rangle, \langle \delta n \delta n \rangle, \langle \delta \varepsilon \delta n \rangle, \ldots$. (Or probability functional).
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2-point (and *n*-point) functions of fluctuating hydro variables: $\langle \delta \varepsilon \delta \varepsilon \rangle$, $\langle \delta n \delta n \rangle$, $\langle \delta \varepsilon \delta n \rangle$, . . . . (Or probability functional).

Relaxation rates of 2pt functions is of the same order as that of corresponding 1pt functions (i.e., $\times 2$).

But effects of fluctuations are *usually suppressed* due to averaging out: $\sqrt{\xi^3/V} \sim (k \xi)^{3/2} \ll 1$ by CLT.

This is why 1st-order hydrodynamics exists (for $d > 2$).
Critical fluctuations

Near CP there is parametric separation of relaxation time scales. The slowest and thus most out-of-equilibrium mode is charge diffusion at const \( p \): \( s/n \equiv m \).

\[
\text{Rate of } m \text{ at scale } \kappa \sim \xi^{-1}, \quad \Gamma \sim D\xi^{-2} \sim \xi^{-3},
\]

Thus we need \( \langle \delta m \delta m \rangle \) as the independent variable(s) in Hydro+ equations.
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is of order of that for sound at much smaller $k \sim \xi^{-3}$.

\[\text{Diagram showing dispersion relations for different modes.}\]
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\[
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\]
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The effect of \( \delta m \) fluctuations is \( (k\xi)^{3/2} = \mathcal{O}(1)! \)

Thus we need \( \langle \delta m \delta m \rangle \) as the independent variable(s) in hydro+ equations.
The new variable is 2-pt function $\langle \delta m \delta m \rangle$ (Wigner transform):

$$\phi_Q(x) = \int_{\Delta x} \langle \delta m(x + \Delta x/2) \delta m(x - \Delta x/2) \rangle e^{iQ \cdot \Delta x}$$

Dependence on $x (\sim L)$ is much slower than on $\Delta x (\sim \xi)$. 

Hydro(+) describes relaxation to equilibrium, maximizing entropy.

To ensure the 2nd law of thermodynamics is obeyed we need to know the entropy:

$s^+(\varepsilon, n, \phi_Q)$, i.e., “EOS+”.

Starting from the definition of $S$ for a given ensemble of states

$$S = \sum_i p_i \log(1/p_i),$$

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\]
Entropy of fluctuations

... a result resembling 2-PI action:

\[ s_{(+)}(\varepsilon, n, \phi_Q) = s(\varepsilon, n) + \frac{1}{2} \int Q \left( 1 - \frac{\phi_Q}{\bar{\phi}_Q} + \log \frac{\phi_Q}{\bar{\phi}_Q} \right) \]
Entropy of fluctuations

... a result resembling 2-PI action:

\[
s_{(+)}(\varepsilon, n, \phi_Q) = s(\varepsilon, n) + \frac{1}{2} \int_Q \left( 1 - \frac{\phi_Q}{\overline{\phi}_Q} + \log \frac{\phi_Q}{\overline{\phi}_Q} \right)
\]

Entropy = log # of states, depends on the width \( \phi_Q \):

- Wider distribution – more microstates
  – more entropy: \( \log (\phi/\overline{\phi})^{1/2} \);
  vs
- Penalty for larger deviations from peak entropy (at \( \delta m = 0 \)): \(- (1/2) \phi/\overline{\phi} \).

Maximum of \( s_{(+)} \) is achieved at \( \phi = \overline{\phi} \).
The equation for $\phi_Q$ is a relaxation equation:

$$(u \cdot \partial) \phi_Q = -\gamma_\pi(Q) \pi_Q, \quad \pi_Q = -\left(\frac{\partial s(+)\varepsilon,n}{\partial \phi_Q}\right)$$

$\gamma_\pi(Q)$ is known from mode-coupling calculation in model H (Kawasaki). It is universal. At $Q \sim \xi^{-1}$, $\gamma_\pi(Q) \sim \xi^{-3}$. 
Hydro+ mode kinetics

The equation for $\phi_Q$ is a relaxation equation:

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$\gamma_\pi(Q)$ is known from mode-coupling calculation in model H (Kawasaki). It is universal. At $Q \sim \xi^{-1}$, $\gamma_\pi(Q) \sim \xi^{-3}$.

The mode distribution function $\phi_Q$ is similar to particle distribution function in kinetic theory (Wigner transform).

In equilibrium, Hydro+ = 1-loop. Similar to kinetic theory vs HTL. Separation of scales: $Q \gg k \sim 1/L$. 

$\Delta p$

$Q + k$

$Q$

$\langle \delta m \delta m \rangle$

$\langle \delta m \delta m \rangle$
Characteristic Hydro-to-Hydro+ crossover rate \( \Gamma_\xi = D\xi^{-2} \sim \xi^{-3} \).

- Dissipation during expansion is overestimated in hydro (---):
  Only modes with \( \omega \ll \Gamma_\xi \) experience large \( \zeta \).

- Stiffness of eos (sound speed) is underestimated in hydro (---):
  Only modes with \( \omega \ll \Gamma_\xi \) are critically soft (\( c_s \rightarrow 0 \) at CP).
A fundamental question about QCD:

*Is there a critical point on the QGP-HG boundary?*

Intriguing results from experiments (BES-I).
More to come from BES-II (also FAIR/CBM, NICA, J-PARC).

Quantitative theoretical framework is needed ⇒ [BEST COLLABORATION].

In H.I.C., the magnitude of the fluctuation signatures of CP is controlled by dynamical *non-equilibrium effects*.

In turn, critical fluctuations affect hydrodynamics.
The interplay of critical and dynamical phenomena: Hydro+. 
More
Critical fluctuations and experimental observables

Observed fluctuations are related to fluctuations of $\sigma$.

[MS-Rajagopal-Shuryak PRD60(1999)114028; MS PRL102(2009)032301]

Think of a collective mode described by field $\sigma$ such that $m = m(\sigma)$:

$$\delta n_p = \delta n_{p}^{\text{free}} + \frac{\partial \langle n_p \rangle}{\partial \sigma} \times \delta \sigma$$

The cumulants of multiplicity $M \equiv \int_p n_p$:

$$\kappa_4[M] = \langle M \rangle + \kappa_4[\sigma] \times g^4 \left( \begin{array}{c} \int_p n_p \gamma_p \\ M^4 \end{array} \right) + \ldots,$$

$$\kappa_4[\sigma] \times g^4 \left( \begin{array}{c} \int_p n_p \gamma_p \\ M^4 \end{array} \right) \sim M^4 \quad \leftarrow \text{acceptance dependent}$$