# Hydro+: Hydrodynamics for QCD critical point

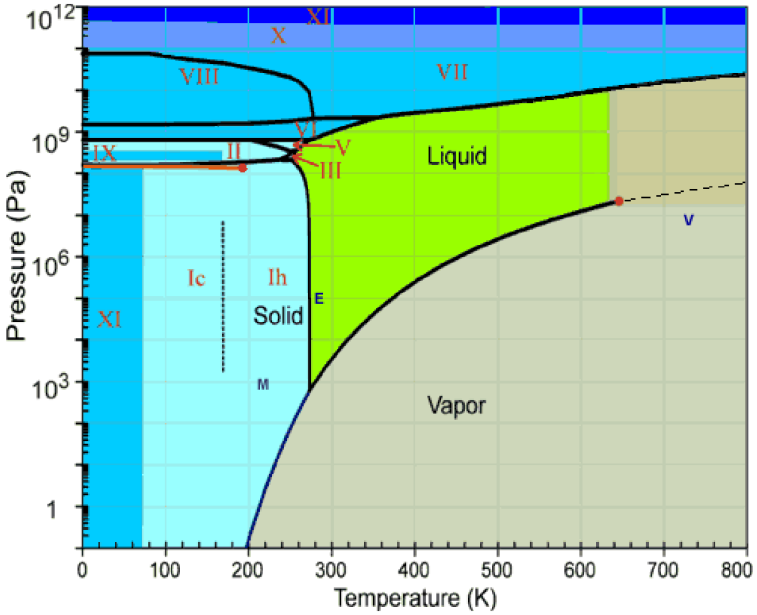
M. Stephanov

with Y. Yin (MIT), 1712.10305

| Substance[13][14] | Critical temperature | Critical pressure (absolute) |
|---|---|---|
| Argon | −122.4 °C (150.8 K) | 48.1 atm (4,870 kPa) |
| Ammonia[15] | 132.4 °C (405.5 K) | 111.3 atm (11,280 kPa) |
| Bromine | 310.8 °C (584.0 K) | 102 atm (10,300 kPa) |
| Caesium | 1,664.85 °C (1,938.00 K) | 94 atm (9,500 kPa) |
| Chlorine | 143.8 °C (416.9 K) | 76.0 atm (7,700 kPa) |
| Ethanol | 241 °C (514 K) | 62.18 atm (6,300 kPa) |
| Fluorine | −128.85 °C (144.30 K) | 51.5 atm (5,220 kPa) |
| Helium | −267.96 °C (5.19 K) | 2.24 atm (227 kPa) |
| Hydrogen | −239.95 °C (33.20 K) | 12.8 atm (1,300 kPa) |
| Krypton | −63.8 °C (209.3 K) | 54.3 atm (5,500 kPa) |
| $CH_4$ (methane) | −82.3 °C (190.8 K) | 45.79 atm (4,640 kPa) |
| Neon | −228.75 °C (44.40 K) | 27.2 atm (2,760 kPa) |
| Nitrogen | −146.9 °C (126.2 K) | 33.5 atm (3,390 kPa) |
| Oxygen | −118.6 °C (154.6 K) | 49.8 atm (5,050 kPa) |
| $CO_2$ | 31.04 °C (304.19 K) | 72.8 atm (7,380 kPa) |
| $N_2O$ | 36.4 °C (309.5 K) | 71.5 atm (7,240 kPa) |
| $H_2SO_4$ | 654 °C (927 K) | 45.4 atm (4,600 kPa) |
| Xenon | 16.6 °C (289.8 K) | 57.6 atm (5,840 kPa) |
| Lithium | 2,950 °C (3,220 K) | 652 atm (66,100 kPa) |
| Mercury | 1,476.9 °C (1,750.1 K) | 1,720 atm (174,000 kPa) |
| Sulfur | 1,040.85 °C (1,314.00 K) | 207 atm (21,000 kPa) |
| Iron | 8,227 °C (8,500 K) | |
| Gold | 6,977 °C (7,250 K) | 5,000 atm (510,000 kPa) |
| Water[2][16] | 373.946 °C (647.096 K) | 217.7 atm (22.06 MPa) |

*Critical point*
– end of phase coexistence –
is a ubiquitous phenomenon

Water:




Is there one in QCD?

# QCD critical point

- QCD is a *relativistic* QFT of a fundamental force,
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- But a critical point is a very universal phenomenon –
  it takes 2 phases whose coexistence (first-order transition) ends.

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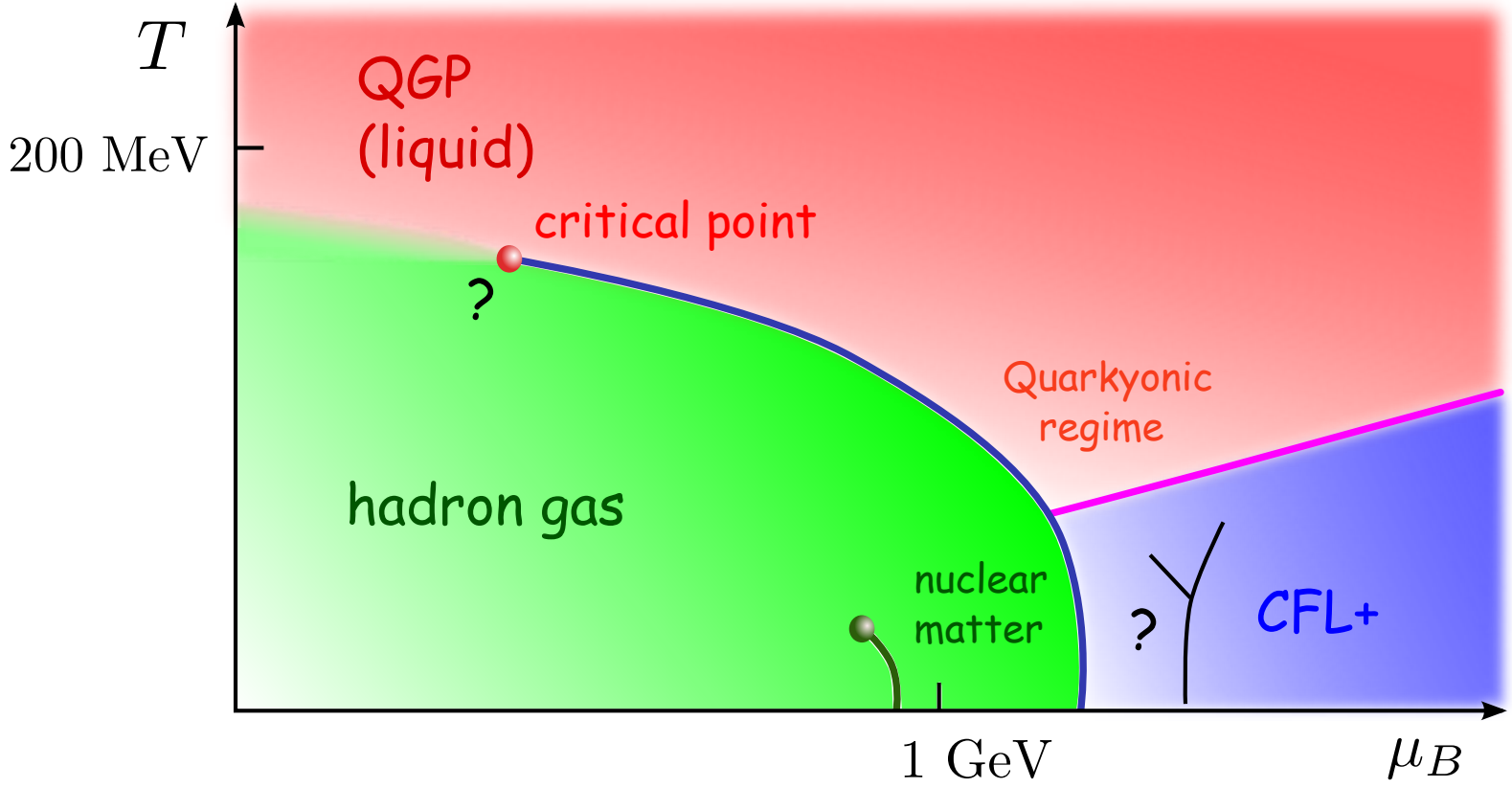
In QCD:

- The two phases: quark-gluon plasma and hadron gas.

  Experiments: QGP has liquid properties – almost perfect fluidity.
- If the phases are separated by a first-order phase transition, there must also be a critical point!

# QCD phase diagram (sketch)



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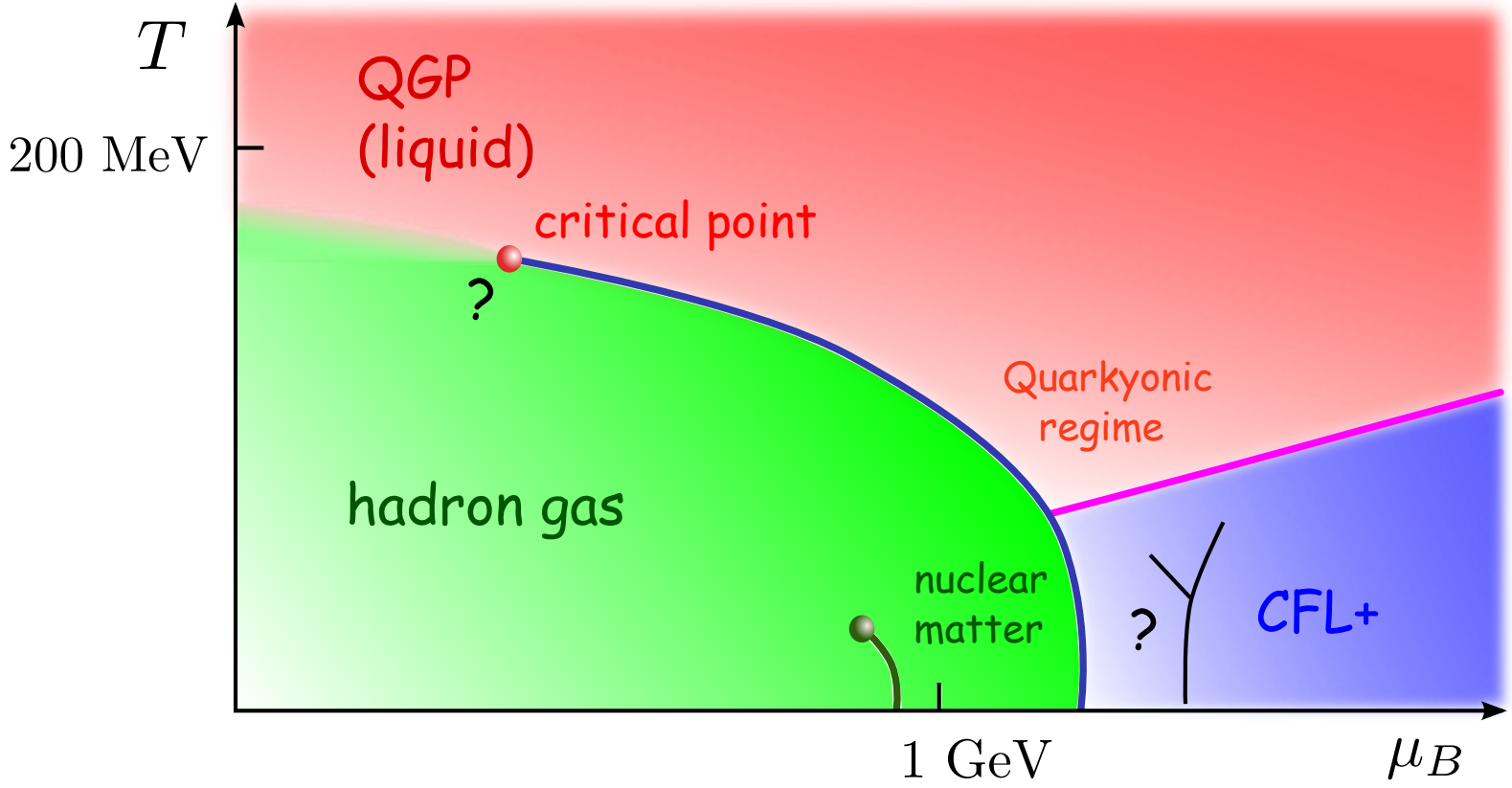


Lattice QCD at $\mu_B \lesssim 2T$ – a crossover

Therefore, if at larger $\mu_B$ $\exists$ first-order transition $\Rightarrow$ $\exists$ critical point

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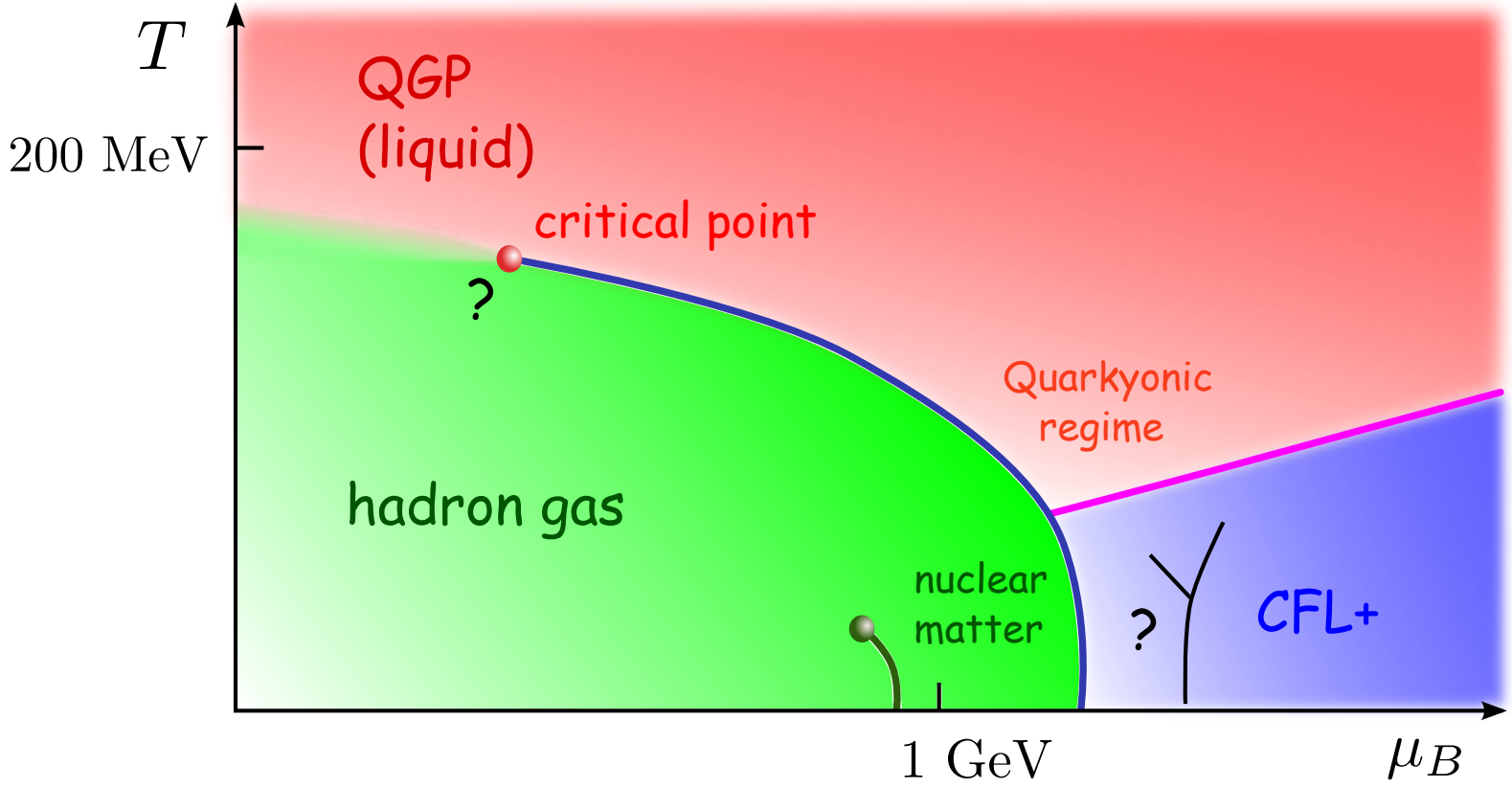


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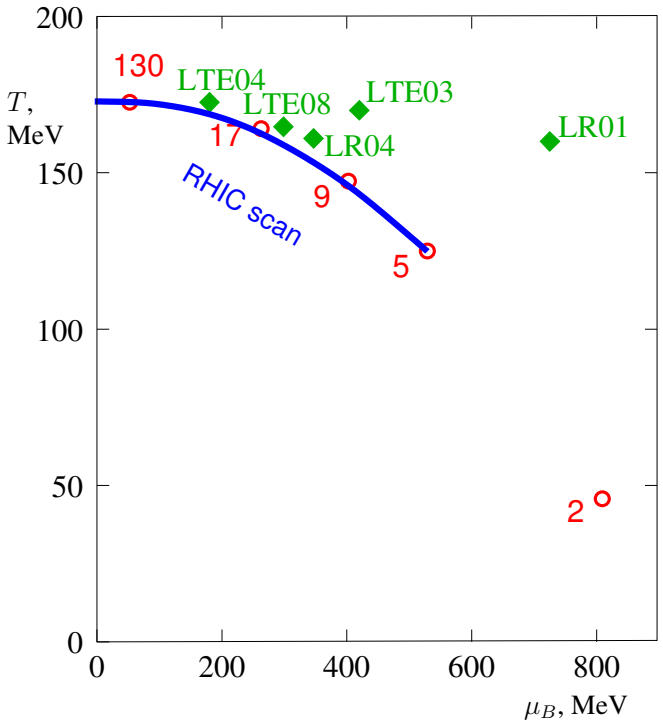
# Critical point discovery challenges

Essentially two approaches to discovering the QCD critical point.

Each with its own challenges.




- Lattice simulations. *Sign problem.*
- Heavy-ion collisions.

  Encouraging progress
  and intriguing new results.

Challenge in connecting the two: *non-equilibrium dynamics.*

# Fluctuations as signatures of the critical point

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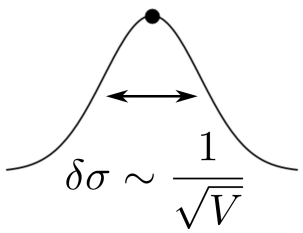

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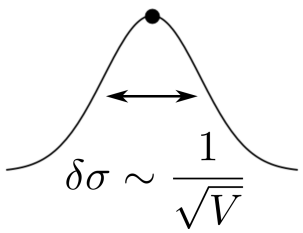
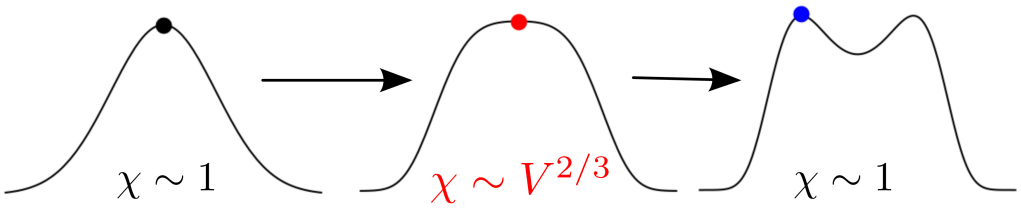



- At the critical point $S(\sigma)$ "flattens". And $\chi \equiv \langle \delta\sigma^2 \rangle V \to \infty$.

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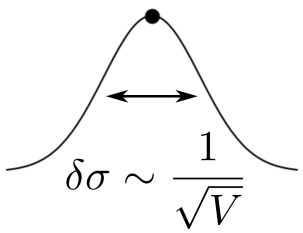


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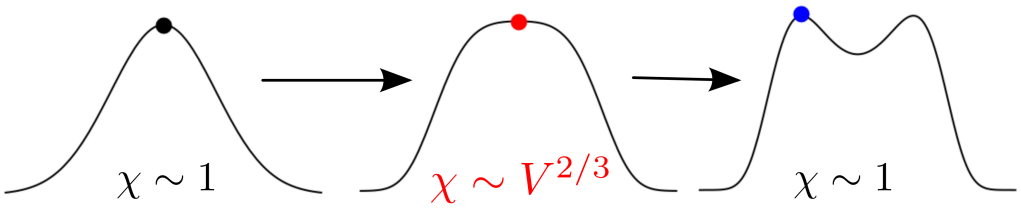


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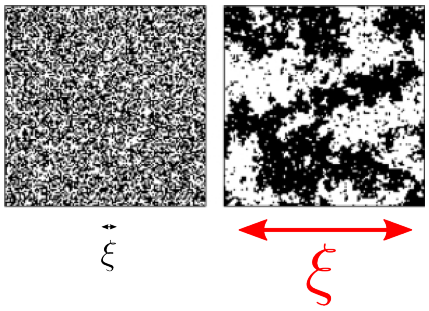



$\delta\sigma$ is not an average of $\infty$ many *uncorrelated* contributions: $\xi \to \infty$

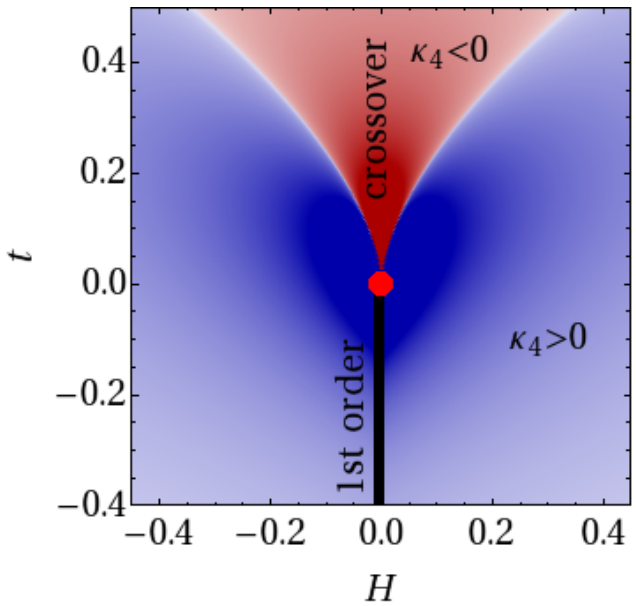
In fact, $\langle \delta\sigma^2 \rangle \sim \xi^2/V$.

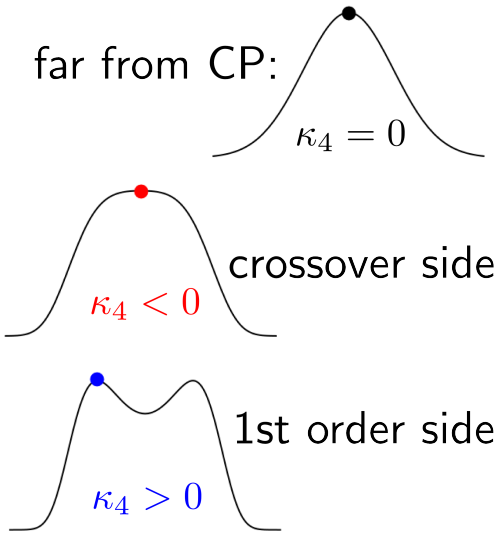
# Higher order cumulants

- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on $\xi$.

  E.g., $\langle \sigma^2 \rangle \sim \xi^2$ while $\kappa_4 = \langle \sigma^4 \rangle_c \sim \xi^7$ *[PRL102(2009)032301]*
- For $n > 2$, sign depends on which side of the CP we are.

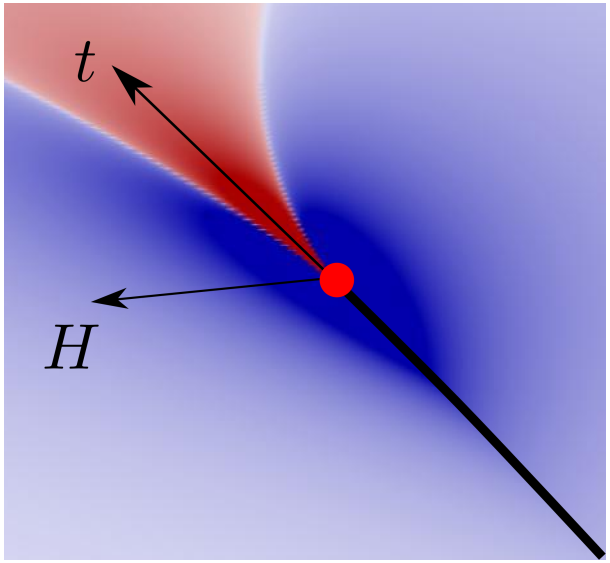
  This dependence is also universal. *[PRL107(2011)052301]*
- Using Ising model variables:



# Mapping Ising to QCD phase diagram

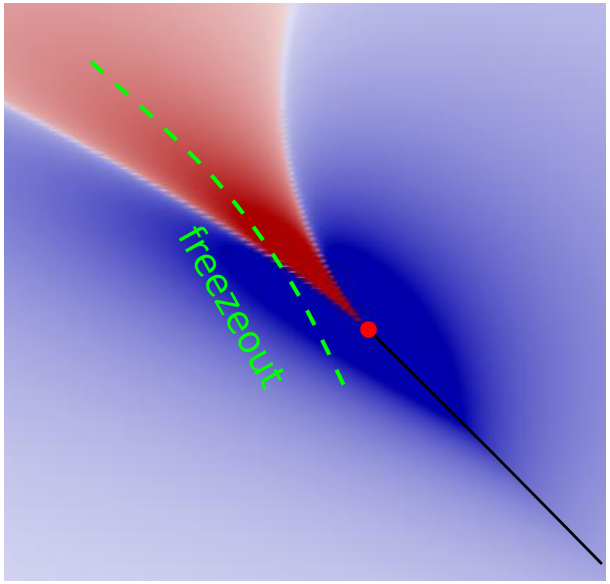
Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:

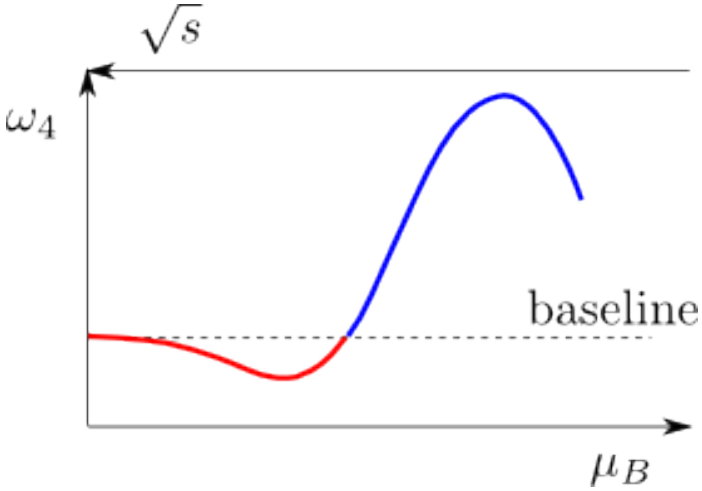



- In QCD $(t, H) \to (\mu - \mu_{\rm CP}, T - T_{\rm CP})$
- $\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$

# Beam Energy Scan I: intriguing hints

Equilibrium $\kappa_4$ vs $T$ and $\mu_B$:



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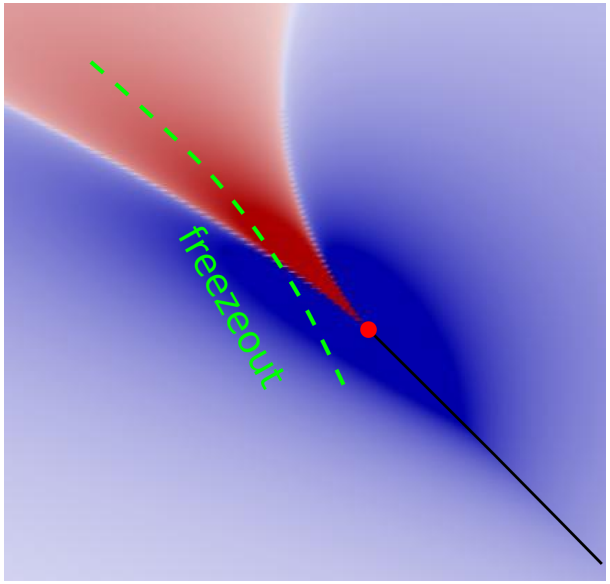


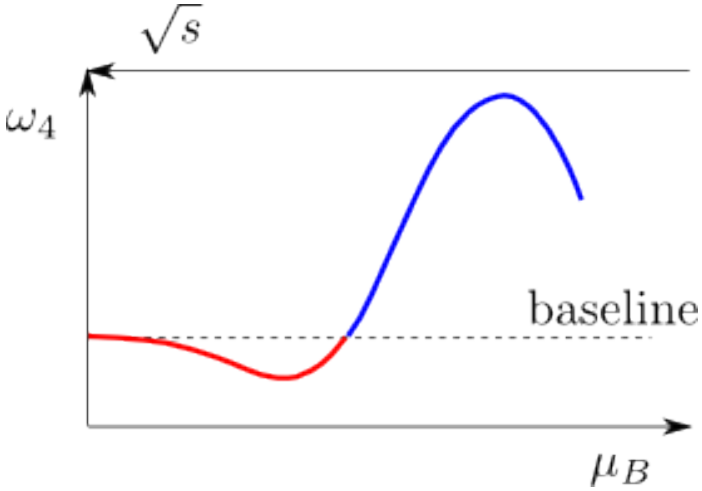


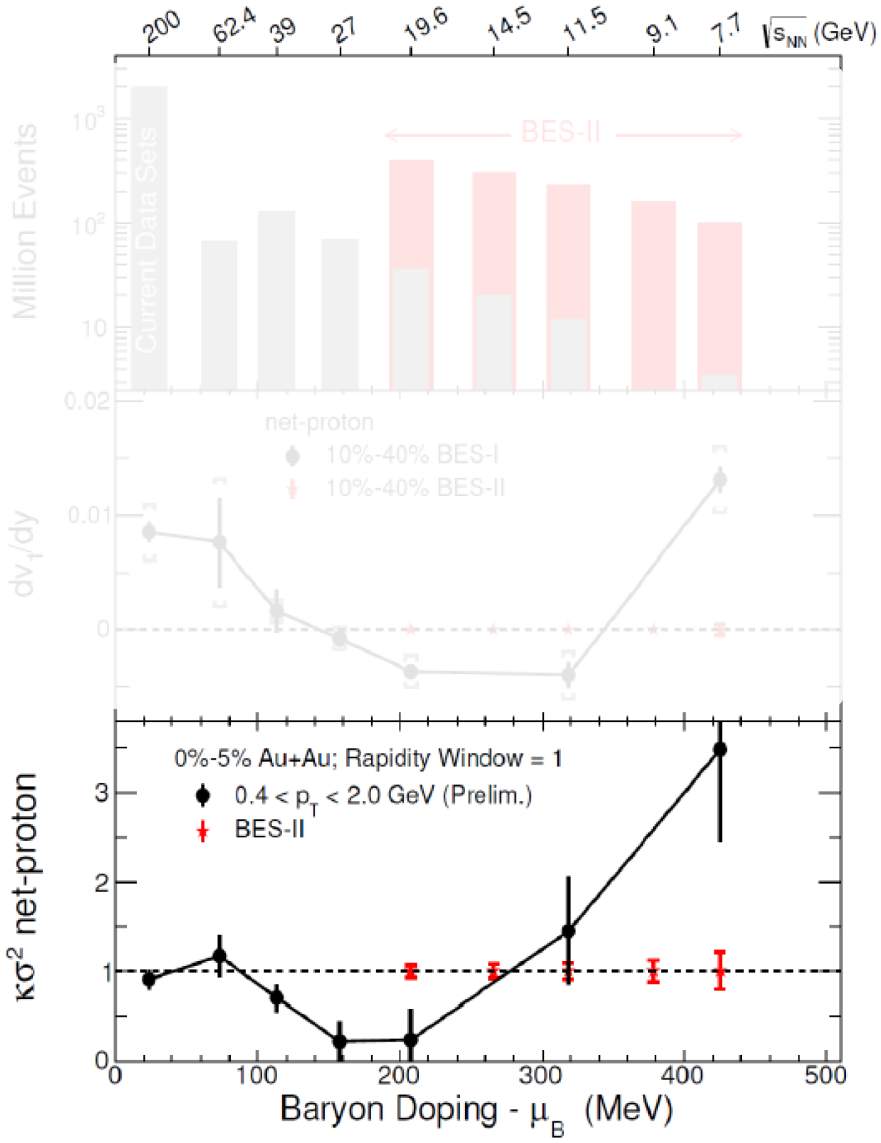

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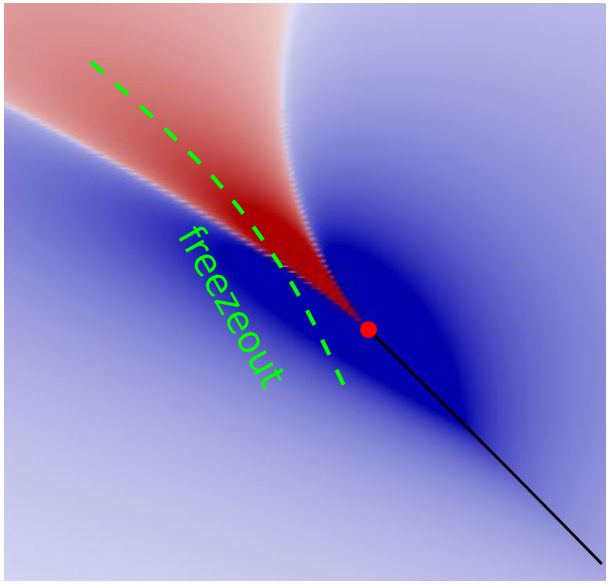
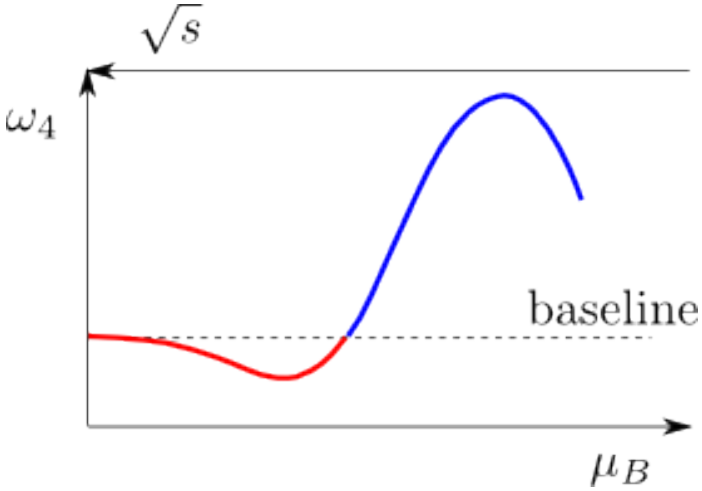
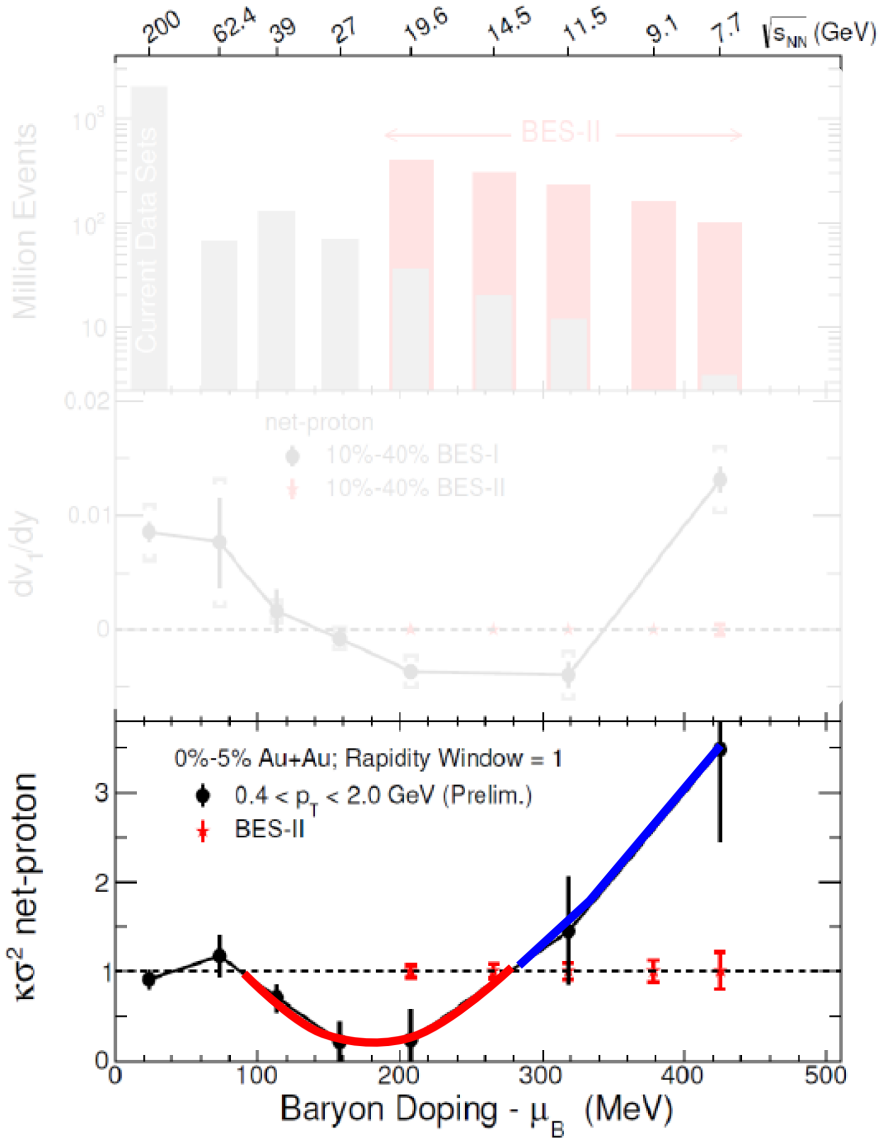



"intriguing hint" (2015 LRPNS)

Non-equilibrium physics is essential near the critical point.

The challenge taken on by BEST COLLABORATION

- Goal: build a *quantitative* theoretical framework describing critical point signatures for comparison with experiment.
- Strategy:
  - Parameterize QCD equation of state with unknown $T_{\mathrm{CP}}$ and $\mu_{\mathrm{CP}}$ as variable parameters.
  - Use it in a hydrodynamic simulation and compare with experiment to determine or constrain $T_{\mathrm{CP}}$ and $\mu_{\mathrm{CP}}$.

# Parameterized EOS for hydro simulations

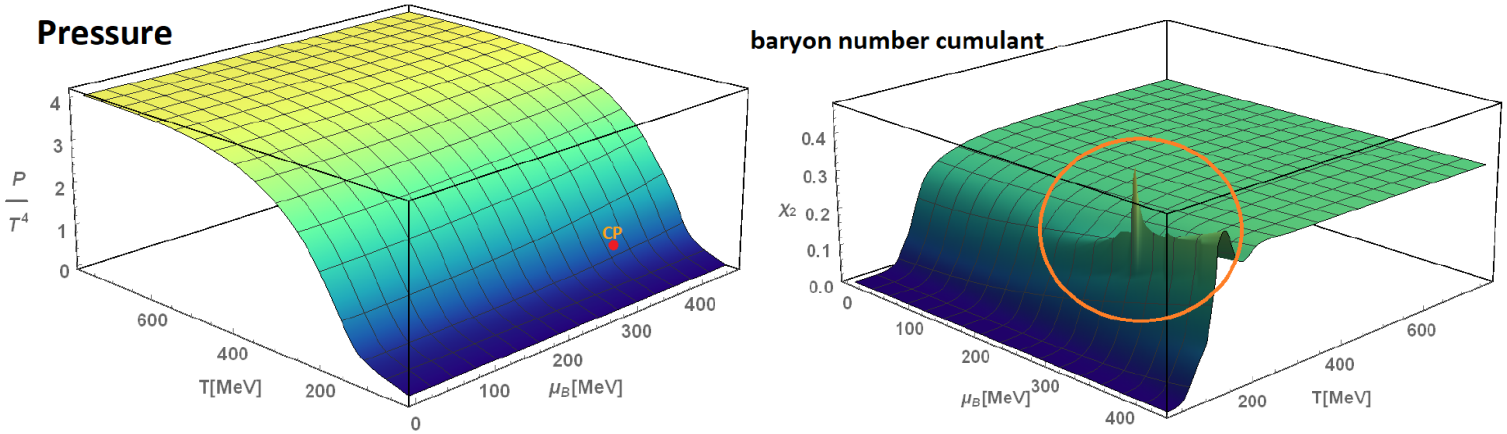
Parotto *et al*, 1805.05249

- Variable parameters ($T_{\rm CP}$, $\mu_{\rm CP}$, slopes, etc.) control Ising-QCD mapping near the QCD critical point: $P = P^{\rm Non\text{-}Ising} + P^{\rm Ising}$.
- Lattice data at $\mu_B = 0$ is matched:

**Decomposition:** Taylor coefficients from Lattice QCD contain an **"Ising"** contribution and a **"Non-Ising"** one:

$$T^4 c_n^{LAT}(T) = T^4 c_n^{\rm Non\text{-}Ising}(T) + T_C^4 c_n^{\rm Ising}(T) \qquad (\bigstar)$$

(parametrization of continuum extrapolated WB Lattice data from S. Borsanyi et al., JHEP 1011 (2010) 077, R. Bellwied et al., Phys.Rev. D92 (2015) no.11, 114505 )




This EOS is ready to be used in a hydrodynamic simulation.

# Hydrodynamics breaks down near the critical point

- Hydrodynamics, as an EFT, relies on separation of scales:

  Evolution rate (e.g., expansion time, $\mathcal{O}(10)$fm) much slower than the local equilibration rate (typically, $\mathcal{O}(0.5-1)$fm).

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- In fact, magnitude of $\xi$, and thus fluctuations/cumulants $\kappa_n \sim \xi^p$, is estimated using $\xi \sim \tau_{\rm expansion}^{1/z}$.

- To be more quantitative we need to describe the breakdown of hydro due to critical slowing down.

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[MS-Yin,1712.10305]

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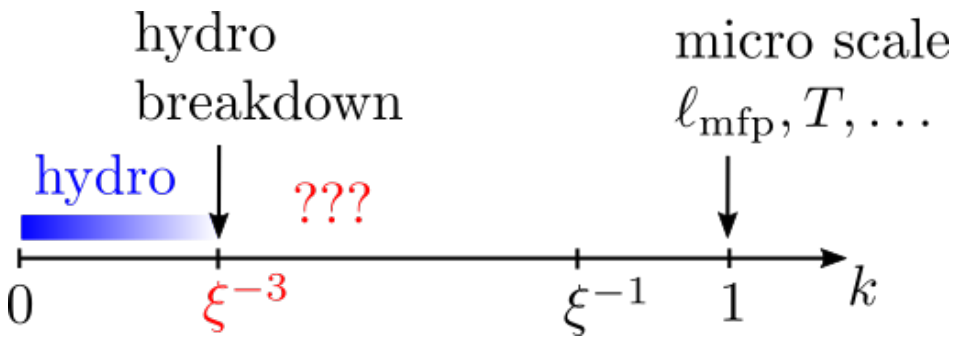
- Breakdown of *locality* is manifested in large gradient corrections to pressure due to $\zeta \sim \xi^3 \to \infty$.

$$p_{\text{hydro}} = p_{\text{equilibrium}} - \zeta \, \boldsymbol{\nabla} \cdot \boldsymbol{v}$$



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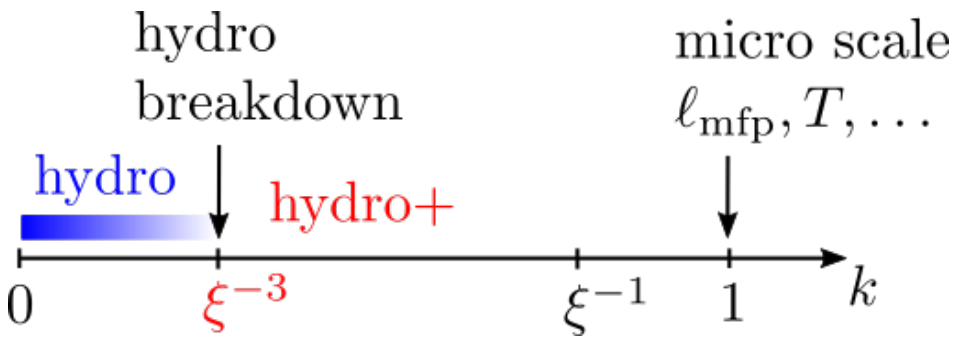


- Extending hydro by adding the critically slow modes $\to$ Hydro+

# What are the additional slow modes?

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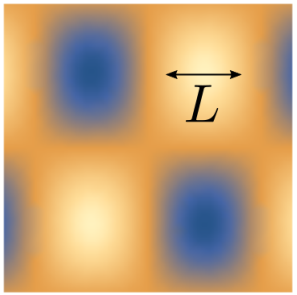
- Hydrodynamics describes *partial-equilibrium states*, i.e., equilibrium is only local, because equilibration time $\sim L^2$.

  *Fluctuations* in such states are not necessarily in equilibrium.



# Nonequilibrium fluctuations

- Measures of fluctuations are *additional* variables needed to characterize the partial-equilibrium state.

  2-point (and $n$-point) functions of fluctuating hydro variables: $\langle \delta\varepsilon\delta\varepsilon \rangle$, $\langle \delta n\delta n \rangle$, $\langle \delta\varepsilon\delta n \rangle$, $\ldots$ . (Or probability functional).

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- Relaxation rates of 2pt functions is of the same order as that of corresponding 1pt functions (i.e., $\times 2$).

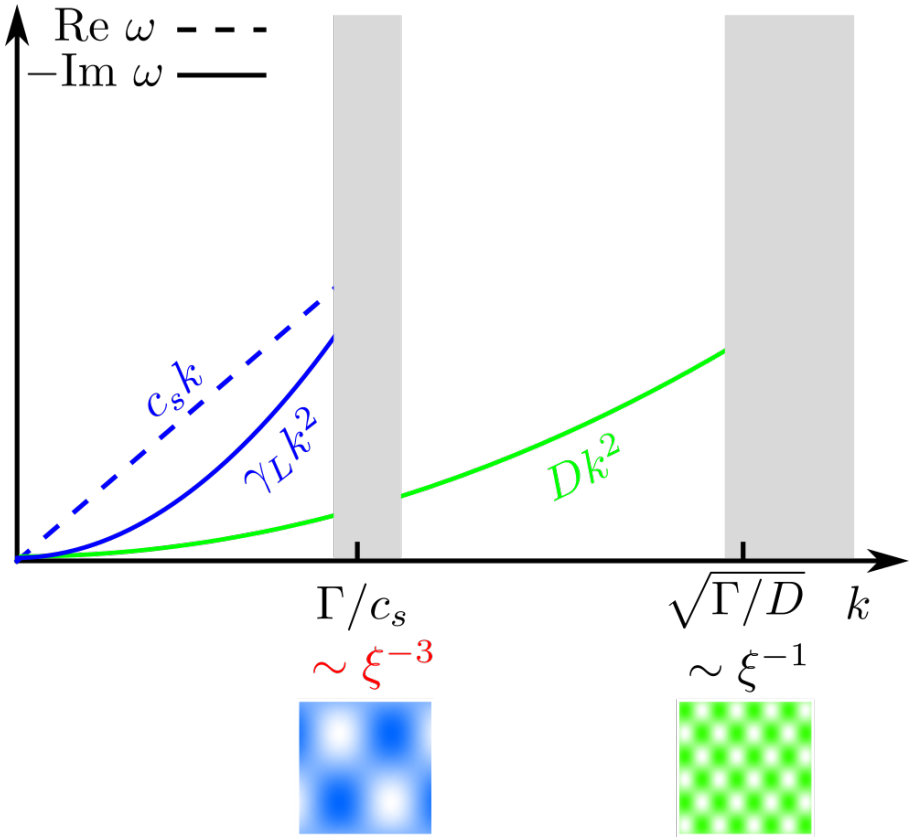
  But effects of fluctuations are *usually suppressed* due to averaging out: $\sqrt{\xi^3/V} \sim (k\xi)^{3/2} \ll 1$ by CLT.

  This is why 1st-order hydrodynamics exists (for $d > 2$).

# Critical fluctuations

- Near CP there is *parametric* separation of relaxation time scales.

  The slowest and thus most out-of-equilibrium mode is charge diffusion at const $p$: $s/n \equiv m$.



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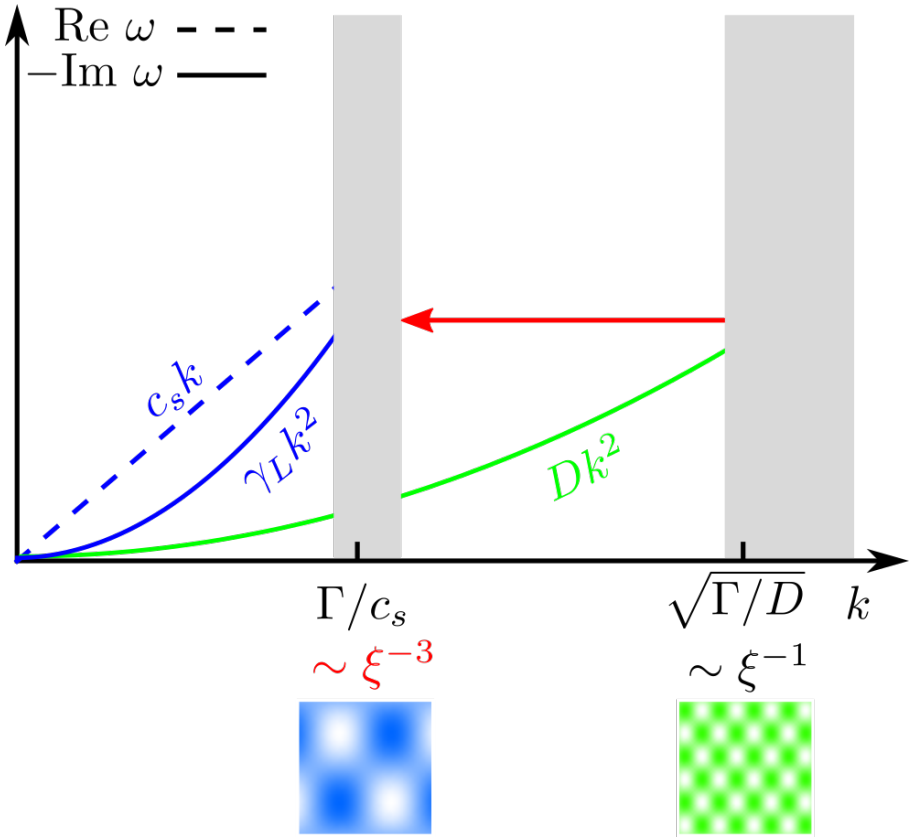

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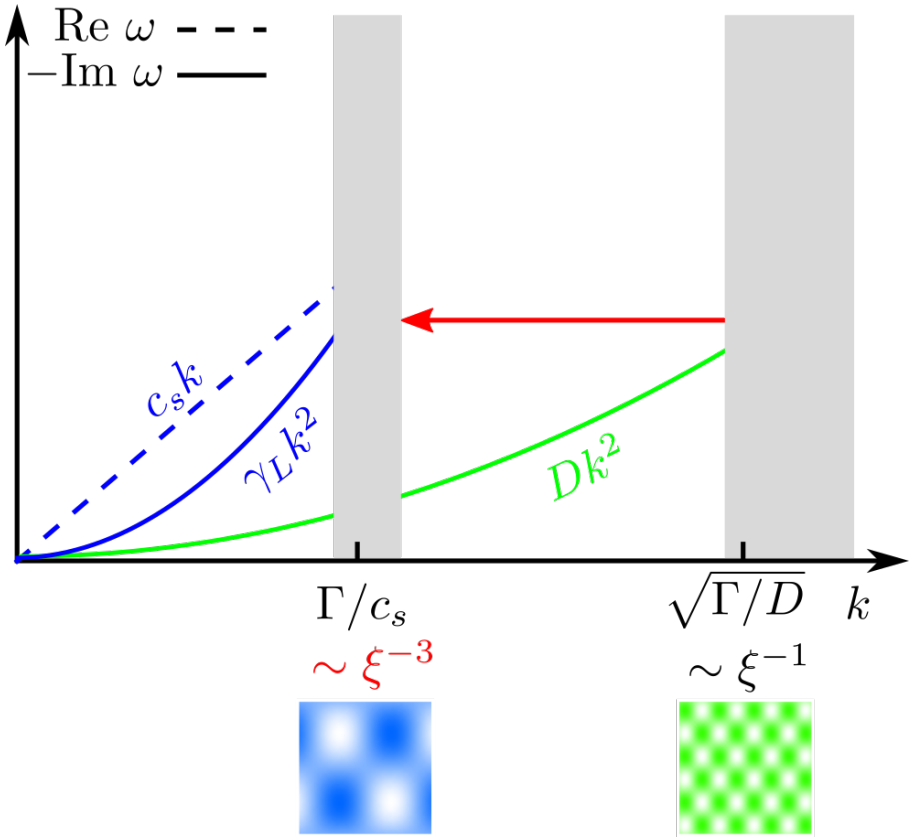

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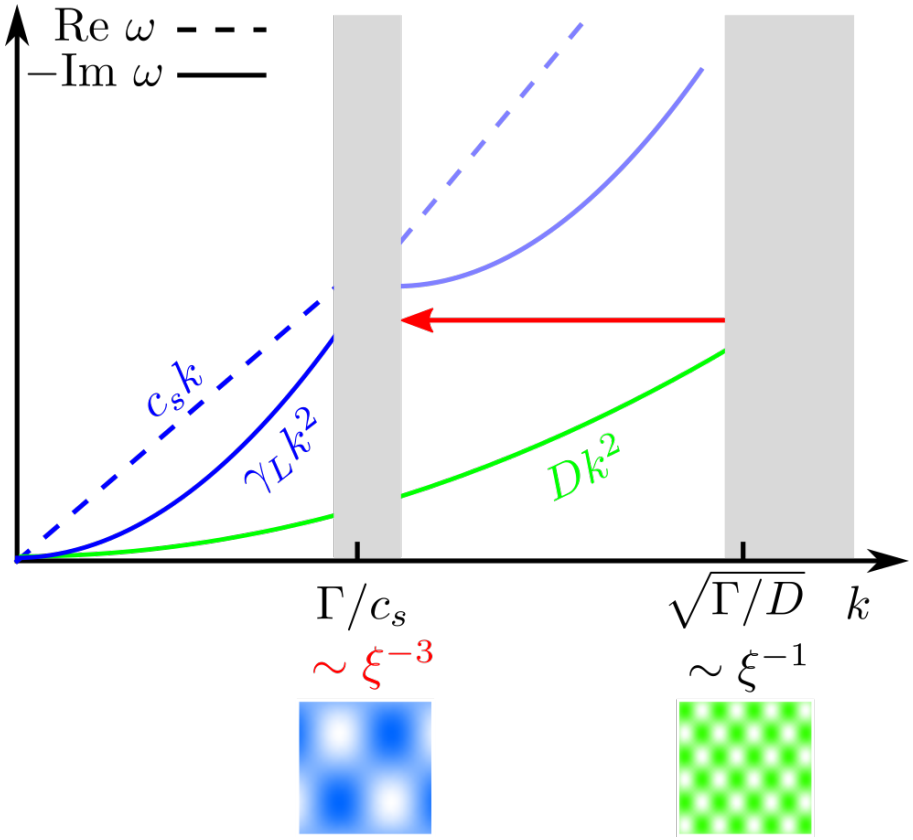
- Thus we need $\langle \delta m \delta m \rangle$ as the independent variable(s) in hydro+ equations.



# New variables and their dynamics

- The new variable is 2-pt function $\langle \delta m \delta m \rangle$ (Wigner transform):

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle\, \delta m(\boldsymbol{x} + \Delta \boldsymbol{x}/2)\, \delta m(\boldsymbol{x} - \Delta \boldsymbol{x}/2)\, \rangle \; e^{i \boldsymbol{Q} \cdot \Delta \boldsymbol{x}}$$

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- Hydro(+) describes relaxation to eqlbrm, maximizing entropy.

  To ensure the 2nd law of thermodynamics is obeyed we need to know the entropy: $s_{(+)}(\varepsilon, n, \phi_{\boldsymbol{Q}})$, i.e., "EOS+".

  Starting from the definition of $S$ for a given ensemble of states

$$S = \sum_i p_i \log(1/p_i), \qquad \text{one arrives at} \ldots$$

# Entropy of fluctuations

- ...a result resembling 2-PI action: *(1712.10305)*

$$s_{(+)}(\varepsilon, n, \phi_{\boldsymbol{Q}}) = s(\varepsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(1 - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + \log\frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}}\right)$$

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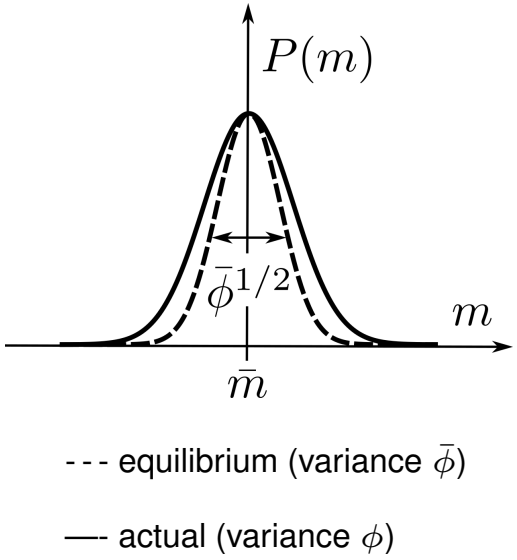
- Entropy = log # of states, depends on the width $\phi_{\boldsymbol{Q}}$:
  - Wider distribution – more microstates – more entropy: $\log(\phi/\bar{\phi})^{1/2}$ ;

    vs
  - Penalty for larger deviations from peak entropy (at $\delta m = 0$): $-(1/2)\phi/\bar{\phi}$.

  Maximimum of $s_{(+)}$ is achieved at $\phi = \bar{\phi}$.



# Hydro+ mode kinetics

- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation:

$$(u \cdot \partial)\phi_{\boldsymbol{Q}} = -\gamma_\pi(\boldsymbol{Q})\pi_{\boldsymbol{Q}}, \quad \pi_{\boldsymbol{Q}} = -\left(\frac{\partial s_{(+)}}{\partial \phi_{\boldsymbol{Q}}}\right)_{\varepsilon,n}$$

$\gamma_\pi(\boldsymbol{Q})$ is known from mode-coupling calculation in model H (Kawasaki). It is universal. At $Q \sim \xi^{-1}$, $\gamma_\pi(Q) \sim \xi^{-3}$.

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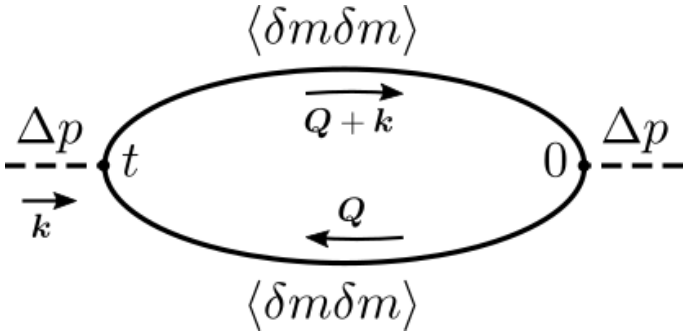
- The mode distribution function $\phi_{\boldsymbol{Q}}$ is similar to particle distribution function in kinetic theory (Wigner transform).

- In equilibrium, Hydro+ = 1-loop.
  Similar to kinetic theory vs HTL.
  Separation of scales: $Q \gg k \sim 1/L$.

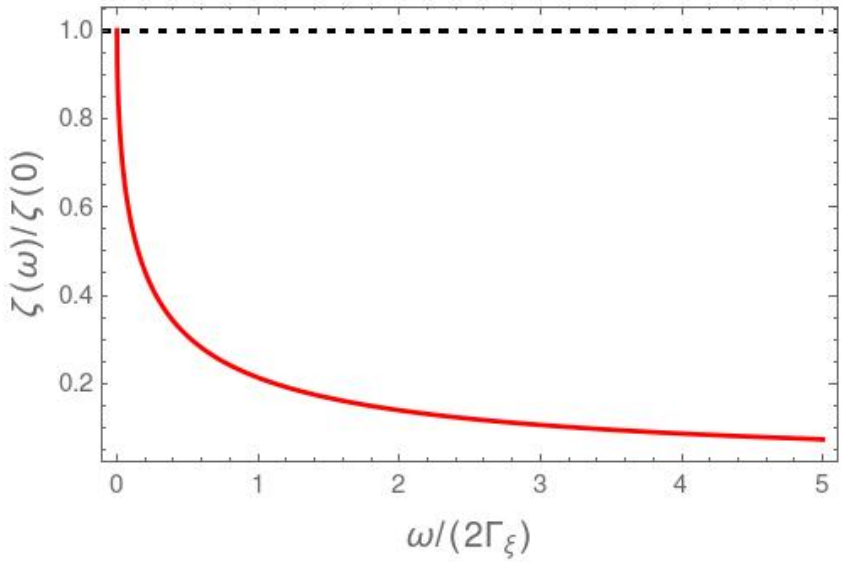


# Hydro+ vs Hydro: real-time bulk response

Characteristic Hydro-to-Hydro+ crossover rate $\Gamma_\xi = D\xi^{-2} \sim \xi^{-3}$.

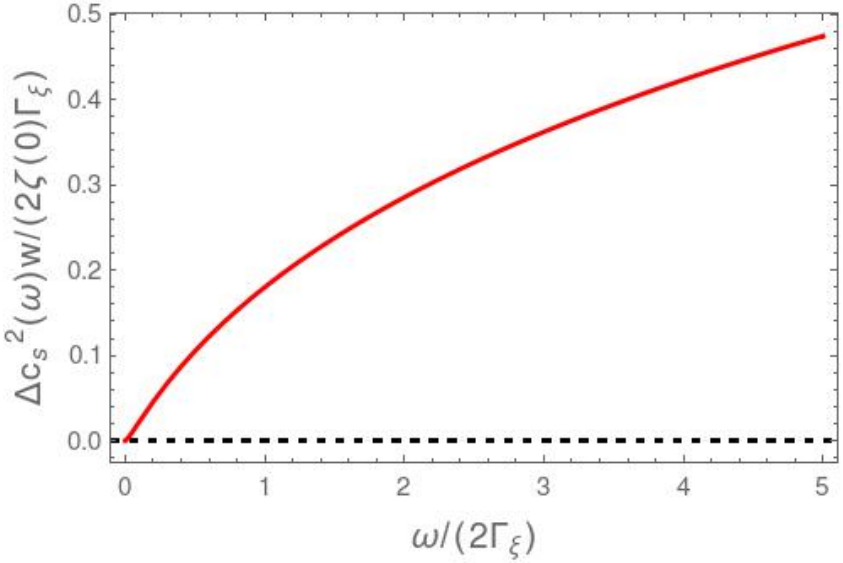
- Dissipation during expansion is overestimated in hydro (- - -):

  Only modes with $\omega \ll \Gamma_\xi$ experience large $\zeta$.




- Stiffness of eos (sound speed) is underestimated in hydro (- - -):

  Only modes with $\omega \ll \Gamma_\xi$ are critically soft ($c_s \to 0$ at CP).



# Summary

- A fundamental question about QCD:

  *Is there a critical point on the QGP-HG boundary?*

- Intriguing results from experiments (BES-I).

  More to come from BES-II (also FAIR/CBM, NICA, J-PARC).

  *Quantitative* theoretical framework is needed $\Rightarrow$ BEST COLLABORATION .

- In H.I.C., the magnitude of the fluctuation signatures of CP is controlled by dynamical *non-equilibrium effects*.

  In turn, critical fluctuations affect hydrodynamics.

  The interplay of critical and dynamical phenomena: Hydro+.

# More

# Critical fluctuations and experimental observables

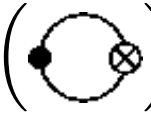
Observed fluctuations are related to fluctuations of $\sigma$.

*[MS-Rajagopal-Shuryak PRD60(1999)114028; MS PRL102(2009)032301]*

Think of a collective mode described by field $\sigma$ such that $m = m(\sigma)$:

$$\delta n_{\boldsymbol{p}} = \delta n_{\boldsymbol{p}}^{\text{free}} + \frac{\partial \langle n_{\boldsymbol{p}} \rangle}{\partial \sigma} \times \delta\sigma$$

The cumulants of multiplicity $M \equiv \int_{\boldsymbol{p}} n_{\boldsymbol{p}}$:

$\kappa_4[M] = \underbrace{\langle M \rangle}_{\text{baseline}} + \kappa_4[\sigma] \times g^4 \underbrace{\Big($  $\Big)^4}_{\sim M^4} + \ldots,$

 $= \int_{\boldsymbol{p}} \frac{n_{\boldsymbol{p}}}{\gamma_{\boldsymbol{p}}}$ $\leftarrow$ acceptance dependent